MATH 10550, EXAM 1 SOLUTIONS

1. If $f(2) = 5$, $f(3) = 2$, $f(4) = 5$, $g(2) = 6$, $g(3) = 2$ and $g(4) = 0$, find $(f \cdot g)(2) + f\big(g(3)\big)$.

Solution. $(f \cdot g)(2) + f\big(g(3)\big) = f(2) \cdot g(2) + f(2) = 5 \cdot 6 + 5 = 35$.

2. Evaluate the following limit

$$\lim_{x \to 0} \frac{2 - \sqrt{4 - x^2}}{x^2}.$$

Solution.

$$\lim_{x \to 0} \frac{2 - \sqrt{4 - x^2}}{x^2} = \lim_{x \to 0} \frac{2 - \sqrt{4 - x^2}}{x^2} \cdot \frac{2 + \sqrt{4 - x^2}}{2 + \sqrt{4 - x^2}} = \lim_{x \to 0} \frac{4 - (4 - x^2)}{x^2(2 + \sqrt{4 - x^2})}$$
$$= \lim_{x \to 0} \frac{1}{2 + \sqrt{4 - x^2}} = \frac{1}{4}.$$

3. For which value of the constant c is the function $f(x)$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} c^2 x - c & x \leq 1 \\ cx - x & x > 1. \end{cases}$$

Solution. The partial functions of $f(x)$ are continuous for $x < 1$ and $x > 1$ because they are polynomials. To get $f(x)$ continuous on $(-\infty, \infty)$ we need

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

This happens when $c^2 - c = c - 1$. Rearranging gives $0 = c^2 - 2c + 1 = (c - 1)^2$, and thus $c = 1$.

4. Compute

$$\lim_{x \to \pi/2+} \tan x.$$

Solution. From the graph of $y = \tan x$, the limit is $-\infty$. Or, since $\tan x = \frac{\sin x}{\cos x}$ and $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, $\tan x$ has a vertical asymptote at $x = \pi/2$. Thus the limit is either ∞ or $-\infty$. For $\pi/2 < x < \pi$, we have $\sin x > 0$ and $\cos x < 0$. Thus for x near $\pi/2$ but greater than $\pi/2$, $\tan x < 0$. Therefore the answer must be $-\infty$.

5. Since the function

$$f(x) = \frac{x^2 - 1}{x^3 - 4x}$$

is a rational function, it is continuous everywhere in its domain, which is everywhere that the denominator is nonzero. The denominator is zero at $x = 0$ and $x = \pm 2$.

6. If $f(x) = (x^2 + 3x)(6x^5 - 2x^8)$, compute $f'(1)$.
Solution. $f'(x) = (2x + 3)(6x^5 - 2x^8) + (x^2 + 3x)(30x^4 - 16x^7)$. $f'(1) = 5 \cdot 4 + 4 \cdot 14 = 76$.

7. For $f(x) = \sqrt[3]{x^5} + \frac{6}{\sqrt[5]{x^3}}$, find $f'(x)$.
Solution. Rewriting $f(x) = x^{\frac{5}{3}} + 6x^{-\frac{3}{5}}$, we have $f'(x) = \frac{5}{3}x^{\frac{2}{3}} + 6(-\frac{3}{5})x^{-\frac{8}{5}} = \frac{5\sqrt[3]{x^2}}{3} - \frac{18}{5\sqrt[5]{x^8}}$.

8. Find the equation of the tangent line to

$$y = \frac{7x - 3}{6x + 2}$$

at the point $(1, \frac{1}{2})$.
Solution.

$$y' = \frac{7(6x + 2) - 6(7x - 3)}{(6x + 2)^2} = \frac{32}{(6x + 2)^2} = \frac{8}{(3x + 1)^2}.$$

Thus, $y'(1) = \frac{1}{2}$ which is the slope of the tangent line at $(1, \frac{1}{2})$. Thus $y = \frac{1}{2}(x - 1) + \frac{1}{2} = \frac{1}{2}x$.

9. If $f(x) = x^2 \cos x$, find $f''(x)$.
Solution. Using Product Rule, we get

$$\begin{aligned} f'(x) &= 2x\cos x - x^2 \sin x, \\ \text{and} \quad f''(x) &= 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x \\ &= 2\cos x - 4x\sin x - x^2\cos x. \end{aligned}$$

10. A ball is thrown straight upward from the ground with the initial velocity $v_0 = 96$ft/s. Find the highest point reached by the ball. Hint: The height of the ball at time t is given by $y(t) = -16t^2 + 96t$.
Solution. Velocity of the ball at time t is given by

$$v(t) = y'(t) = -32t + 96.$$

The ball reaches the highest point when $v(t) = 0$, i.e. when $t = 3$ seconds. The height of the ball at 3 seconds is

$$y(3) = -16(3)^2 + 96(3) = -144 + 288 \text{ ft.} = 144 \text{ ft.}$$

11. Find the equation of the tangent line to the curve $y = \frac{x^3}{3} - x^2 + 1$ which is parallel to the line $y + x = 4$.

Solution. The line parallel to the line $y + x = 4$ will have the same slope, namely -1. So we need to find the point on the curve which has slope -1. $y' = x^2 - 2x$. We solve for x given $y' = -1$:

$$x^2 - 2x = -1 \Longrightarrow (x-1)(x-1) = 0 \Longrightarrow x = 1.$$

Plugging into the equation for the curve we see that $y = 1/3$ at this point. The tangent line at $(1, \frac{1}{3})$ is given by

$$y - \frac{1}{3} = -(x-1), \qquad \text{or} \qquad y = -x + \frac{4}{3}.$$

12. Show that there are at least *two* roots of the equation

$$x^4 + 6x - 2 = 0.$$

Justify your answer and identify the theorem you use.

Solution. Let $f(x) = x^4 + 6x - 2$. Then $f(-2) = 2$, $f(0) = -2$ and $f(1) = 5$. Since $f(x)$ is a polynomial, f is continuous on the real line. We have $f(-2) > 0 > f(0)$. So, by the **Intermediate Value Theorem**, there exists a number c between -2 and 0 such that $f(c) = 0$. Similarly, there exists a number d between 0 and 1 such that $f(d) = 0$.

Note: The choices $x = -2, 0, 1$ are not the only possibilities.

13. Given

$$y = \frac{1}{x^2 + 1},$$

find y' using the **definition** of the derivative.

Solution.

$$\begin{aligned}
\text{Let } f(x) &= \frac{1}{x^2+1}. \\
\text{Then } y' = f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(x^2+1)-((x+h)^2+1)}{((x+h)^2+1)\cdot(x^2+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{1} - \cancel{x^2} - 2xh - h^2 - \cancel{1}}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{-2x-h}{((x+h)^2+1)(x^2+1)} \\
&= \frac{-2x-0}{((x+0)^2+1)(x^2+1)} \\
&= -\frac{2x}{(x^2+1)^2}.
\end{aligned}$$