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Example Alan, Cassie, Maggie, Seth and Roger are friends who want to take a photograph with three of the five friends in it.

Alan (who likes to be thorough) makes a complete list of all possible ways of lining up 3 out of the 5 friends for a photo as follows:

Section 6.4: Permutations

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

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He reckons he has 5 choices for the position on the left, and

once he's chosen who should stand on the left, he will have 4 choices for the position in the middle

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This gives a total of $5 \times 4 \times 3 = 60$ possibilities.

Section 6.4: Permutations

Alan has listed all **Permutations** of the five friends taken 3 at a time.

The number of permutations of 5 objects taken 3 at a time has a special symbol:

$$\mathbf{P}(5,3)$$

and as we have seen $\mathbf{P}(5,3) = 60$.

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Definition A **Permutation** of **n** objects taken **k** at a time is an arrangement (Line up, Photo) of k of the n objects in a specific <u>order</u>. The symbol for this number is $\mathbf{P}(\mathbf{n}, \mathbf{k})$.

Section 6.4: Permutations

When using the multiplication principle to count the number of such permutations, as Alan did, the following characteristics are key:

1. A permutation is an arrangement or sequence of selections of objects from a single set.
2. Repetitions are not allowed. Equivalently the same element may not appear more than once in an arrangement. (In the example above, the photo AAA is not possible).
3. the order in which the elements are selected or arranged is significant. (In the above example, the photographs AMC and CAM are different).

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Example Calculate $\mathbf{P}(6, 4)$, the number of photographs of 6 friends taken 4 at a time.

$\mathbf{P}(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360.$

Section 6.4: Permutations

We can use the same principles that Alan did to find a general formula for the number of permutations of n objects taken k at a time, which follows from an application of the multiplication principle:

$$\boxed{\mathbf{P}(n,k) = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1).}$$

Note that there are k consecutive numbers on the right hand side.

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Note that there are k consecutive numbers on the right hand side.

Some of you may have buttons on your calculators that will compute $\mathbf{P}(n,k)$. Check the manual to see how to do this.

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Example In how many ways can you choose a President, secretary and treasurer for a club from 12 candidates, if each candidate is eligible for each position, but no candidate can hold 2 positions? Why are conditions 1, 2 and 3 relevant here?

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Condition 1 is satisfied because we have a single set of 12 candidates for all 3 positions.
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$\mathbf{P}(30,10) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21.$ Note $30 - 10 = 20$ and we stopped at 21.

$\mathbf{P}(30,10) = 109,027,350,432,000$

Section 6.4: Permutations

Example In how many ways can you arrange 5 math books on a shelf.

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$\mathbf{P}(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Note $5 - 5 = 0$ and we stopped at 1.

$\mathbf{P}(5,5) = 120$

Factorials

The numbers $\mathbf{P}(n,n) = n \cdot (n-1) \cdot (n-2) \cdots 1$ are denoted by $n!$ or factorial n. We can rewrite our formula for $\mathbf{P}(n,k)$ in terms of factorials:

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Example (a) Evaluate 12!

(b) Evaluate $\mathbf{P}(12,5)$.

$12! = \mathbf{P}(12,12) = 12 \cdot 11 \cdots 2 \cdot 1 = 479,001,600.$

$\mathbf{P}(12,5) = \frac{\mathbf{P}(12,12)}{\mathbf{P}(7,7)} = \frac{479,001,600}{5,040} = 95,040.$

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$\mathbf{P}(26, 3) = 15, 600.$

Permutations of objects with some alike

In this section, we will only consider permutations of collections of n objects taken n at at time, in other words rearrangements of n objects.

We will consider situations in which some objects are the same. Note that if two objects in the arrangement are the same, we get the same arrangement when we switch the two.

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BEER?

The set $\{B, E, E, R\} = \{B, E, R\}$ but we really have 4 letters with which to work. So let us start with the set $\{B, R, \color{blue}{E}, \color{red}{E}\}$. We arrange them in $4! = 24$ ways:

B R $\color{blue}{E}$ $\color{red}{E}$	B $\color{blue}{E}$ R $\color{red}{E}$	B $\color{blue}{E}$ $\color{red}{E}$ R	R B $\color{blue}{E}$ $\color{red}{E}$	R $\color{blue}{E}$ B $\color{red}{E}$	R $\color{blue}{E}$ $\color{red}{E}$ B	$\color{blue}{E}$ B R $\color{red}{E}$	$\color{blue}{E}$ B $\color{red}{E}$ R	$\color{blue}{E}$ $\color{red}{E}$ B R	$\color{blue}{E}$ R B $\color{red}{E}$	$\color{blue}{E}$ R $\color{red}{E}$ B	$\color{blue}{E}$ $\color{red}{E}$ R B
B R $\color{red}{E}$ $\color{blue}{E}$	B $\color{red}{E}$ R $\color{blue}{E}$	B $\color{red}{E}$ $\color{blue}{E}$ R	R B $\color{red}{E}$ $\color{blue}{E}$	R $\color{red}{E}$ B $\color{blue}{E}$	R $\color{red}{E}$ $\color{blue}{E}$ B	$\color{red}{E}$ B R $\color{blue}{E}$	$\color{red}{E}$ B $\color{blue}{E}$ R	$\color{red}{E}$ $\color{blue}{E}$ B R	$\color{red}{E}$ R B $\color{blue}{E}$	$\color{red}{E}$ R $\color{blue}{E}$ B	$\color{red}{E}$ $\color{blue}{E}$ R B

Permutations of objects with some alike

B R E E	B E R E	B E E R	R B E E	R E B E	R E E B	E B R E	E B E R	E E B R	E R B E	E R E B	E E R B
B R E E	B E R E	B E E R	R B E E	R E B E	R E E B	E B R E	E B E R	E E B R	E R B E	E R E B	E E R B

If we just want the words, each entry in the top row gives the same word as the entry below it on the bottom row. In other words, if we switch the two E's in any arrangement, we do not get a new word.
So if we count all permutations of 4 letters, we over count the number of words. Thus among the $4! = 24$ arrangements of the 4 letters above, the word EEBR appears twice. Similarly every other word appears twice on the list of 4! arrangements. Thus the number of different words we can form by rearranging the letters must be

$$4!/2 = \frac{4!}{2!}$$

Note that 2! counts the number of ways we can permute the E's in any given arrangement.

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We can see this as follows. We have n positions to fill. $\mathbf{P}(n, n-r)$ is the number of ways to put the $n-r$ elements which are unique into the n positions. Once we have done this we just fill in the remaining positions with the repeated element.

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From our previous example:

B R _ _	B _ R _	B _ _ R	R B _ _	R _ B _	R _ _ B	_ B R _	_ B _ R	_ _ B R	_ R B _	_ R _ B	_ _ R B

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Example How many distinct words(including nonsense words) can be made from rearrangements of the word

ALPACA

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$\dfrac{6!}{3!}$. There are 6 letters in ALPACA and one of them, 'A' is repeated 3 times. $\frac{6!}{3!} = \frac{720}{6} = 120$

Permutations of objects with some alike

Suppose given a collection of n objects containing k subsets of objects in which the objects in each subset are identical and objects in different subsets are not identical. Then the number of different permutations of all n objects is

$$\frac{n!}{r_1! \cdot r_2! \ \cdots r_k!},$$

where r_1 is the number of objects in the first subset, r_2 is the number of objects in the second subset, etc.

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Note that if you make the collection of objects into a set, the set has k elements in it.
Note that for a subset of size 1, we have $1! = 1$, so this formula is a generalization of the previous one.

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Permutations of objects with some alike

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$\{B, A, N, A, N, A\} = \{A, B, N\}$ or $k = 3.$ There are 6 letters in BANANA.
The 'A' is repeated 3 times.
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Hence the answer is $\dfrac{6!}{1! \cdot 2! \cdot 3!}$.

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The 'N' is repeated 2 times.
The 'B' is repeated once.

As a safety check, note that $1 + 2 + 3 = 6$.

Permutations of objects with some alike

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BOOKKEEPER

Permutations of objects with some alike

Example How many distinct words(including nonsense words) can be made from rearrangements of the word

BOOKKEEPER

$$\frac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!}.$$

There are 10 letters in BOOKKEEPER. In alphabetical order, B $\leftrightarrow$ 1, E $\leftrightarrow$ 3, K $\leftrightarrow$ 2, O $\leftrightarrow$ 2, P $\leftrightarrow$ 1, R $\leftrightarrow$ 1.

Note that the total number of letters is the sum of the multiplicities of the distinct letters.

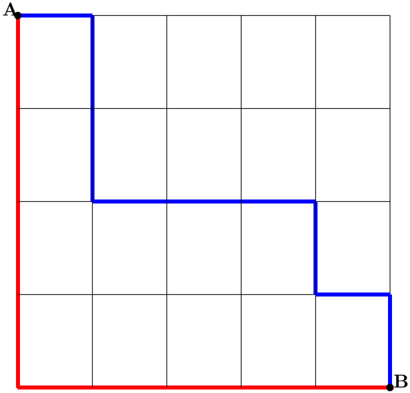
$\frac{10!}{1!\cdot 3!\cdot 2!\cdot 2!\cdot 1!\cdot 1!} = \frac{3{,}628{,}800}{6\cdot 2\cdot 2} = 151,200.$

Taxi cab Geometry

On the street map shown below, any route that a taxi cab can take from the point A to the point B if they always travel south or east can be described uniquely as a sequence of S's and E's (S for South and E for East). To get from A to B the taxi driver must travel south for four blocks and east for five blocks. Any sequence of 4 S's and 5 E's describes such a route and two routes are the same only if the sequences describing them are the same. Thus the number of taxi cab routes from A to B is the number of different rearrangements of the sequence SSSSEEEEE which is $\dfrac{9!}{4!5!}$.

Taxi cab Geometry

Here we show the sequence SSSSEEEEE in red and the sequence ESSEEEESES in blue.



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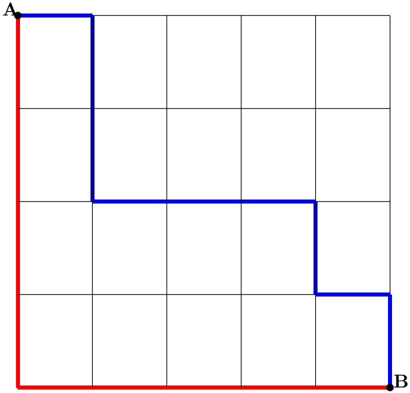
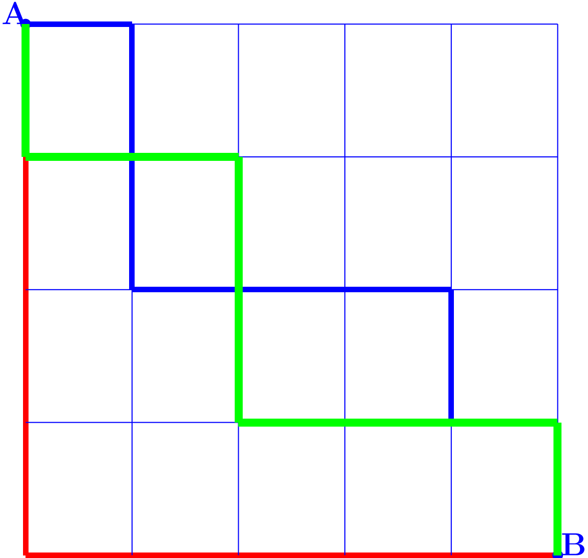


Draw the sequence SEESSEEES.

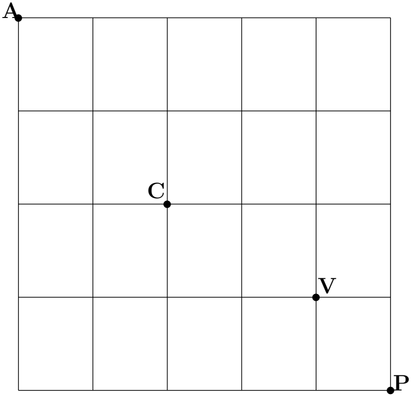
Taxi cab Geometry

SEESSEEES is drawn in green.

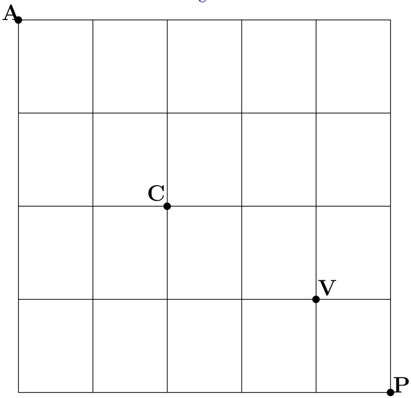


Taxi cab Geometry

Example A streetmap of Mathville is given below. You arrive at the Airport at A and wish to take a taxi to Pascal's house at P. The taxi driver, being an honest sort, will take a route from A to P with no backtracking, always traveling south or east.

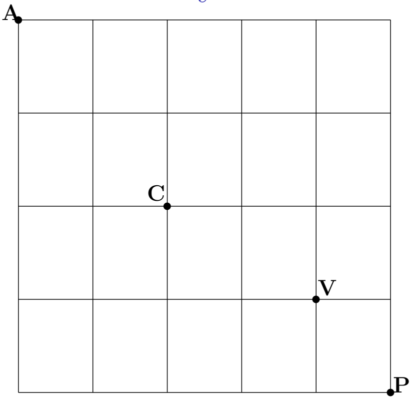


Taxi cab Geometry



(a) How many such routes are possible from A to P?

Taxi cab Geometry

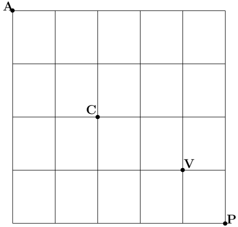


(a) How many such routes are possible from A to P?

You need to go 4 blocks south and 5 blocks east for a total of 9 blocks so the number of routes is

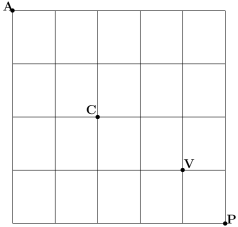
$$\frac{9!}{4! \cdot 5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 2 \cdot 7 = 126.$$

Taxi cab Geometry



(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

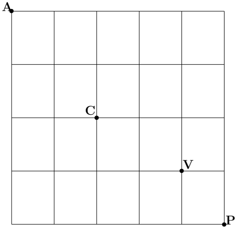
Taxi cab Geometry



(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

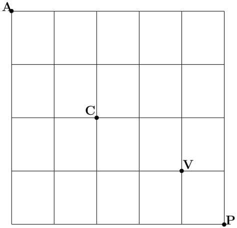
This is really two taxicab problems combined with the Multiplication Principle. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to P'. The first is $\dfrac{4!}{2! \cdot 2!} = 6$ and the second is $\dfrac{5!}{2! \cdot 3!} = 10$ so the answer is $6 \cdot 10 = 60$.

Taxi cab Geometry



(c) If wish to stop off at both the combinatorium at C and the Vennitarium at V, how many routes can your taxi driver take?

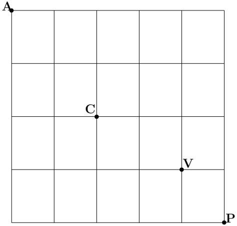
Taxi cab Geometry



(c) If wish to stop off at both the combinatorium at C and the Vennitarium at V, how many routes can your taxi driver take?

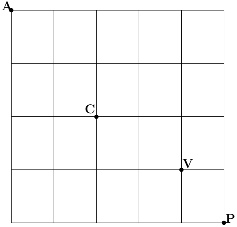
This is three taxicab problem. The answer, in words, is 'the number of paths from A to C' times 'the number of paths from C to V' times 'the number of paths from V to P. The first is $\dfrac{4!}{2! \cdot 2!} = 6$, the second is $\dfrac{3!}{1! \cdot 2!} = 3$ and the third is $\dfrac{2!}{1! \cdot 1!} = 2$ so the answer is $6 \cdot 3 \cdot 2 = 36$.

Taxi cab Geometry



(d) If you wish to stop off at either C or V(at least one), how many routes can the taxi driver take.

Taxi cab Geometry

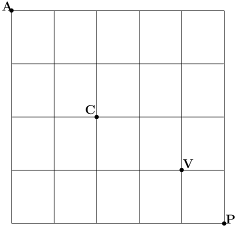


(d) If you wish to stop off at either C or V(at least one), how many routes can the taxi driver take.

This certainly the most complicated of this set of problems. It involves not only taxis but also the Inclusion-Exclusion Principle. To see this, suppose C denotes the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

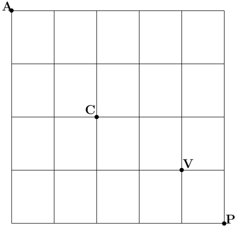
Taxi cab Geometry



Suppose C denotes the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

Taxi cab Geometry



Suppose C denotes the set of all paths from A to P that go through C and that V denotes the set of all paths from A to P that go through V.

The number we want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

We have already computed $n(C) = 60$. For $n(V)$ we have

$$n(V) = \frac{7!}{3! \cdot 4!} \cdot \frac{2!}{1! \cdot 1!} = \frac{7 \cdot 6 \cdot 5}{6} \cdot 2 = 70.$$

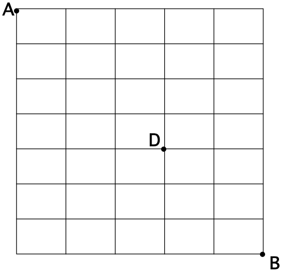
We still need $n(C \cap V)$ but $C \cap V$ is the set of all paths which go through both C and V and we have already computed this: $n(C \cap V) = 36$.

Hence

$$n(C \cup V) = 60 + 70 - 36 = 94$$

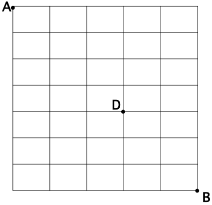
Taxi cab Geometry

Example Christine, on her morning run, wants to get from point A to point B.



(a)How many routes with no backtracking can she take?
(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?

Taxi cab Geometry



(a)How many routes with no backtracking can she take?
(b) How many of those routes go through the point D?
(c) If Christine wants to avoid the Doberman at D, how many routes can she take?

(a) $\dfrac{(5+7)!}{5! \cdot 7!}$

(b) $\dfrac{(3+4)!}{3! \cdot 4!} \cdot \dfrac{(2+3)!}{2! \cdot 3!}$

(c) $\dfrac{(5+7)!}{5! \cdot 7!} - \left(\dfrac{(3+4)!}{3! \cdot 4!} \cdot \dfrac{(2+3)!}{2! \cdot 3!} \right)$