Combinations

Example Recall our five friends, Alan, Cassie, Maggie, Seth and Roger from the example at the beginning of the previous section. They have won 3 tickets for a concert in Chicago and everybody would like to go. However, they cannot afford two more tickets and must choose a group of three people from the five to go to the concert.

Combinations

Example Recall our five friends, Alan, Cassie, Maggie, Seth and Roger from the example at the beginning of the previous section. They have won 3 tickets for a concert in Chicago and everybody would like to go. However, they cannot afford two more tickets and must choose a group of three people from the five to go to the concert.

Alan decides to make a fair decision on who gets the tickets, by writing the initials of each possible group of 3 individuals from the 5 friends on a separate piece of paper and putting all of the pieces of paper in a hat. He will then draw one piece of paper from the hat at random to select the lucky group. The order in which the names of the individuals is written is irrelevant. The complete list of the 10 possible groups of size three from the five friends is shown below:

Combinations

AMC	AMS	AMR	ACS	ACR
ASR	MSR	MCR	MCS	CRS

Note that for every group on this list, it appears $6 = 3!$ times on the list of possible photographs of three of the five friends that we saw in the last lecture:

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

Combinations

Its easy to see that the reason for this is because every group of three individuals can be permuted in $3! = 6$ ways to create $3!$ different photographs. When selecting a group from the hat to receive the tickets, the order in which the names of the individuals appear on the slip of paper is irrelevant. The group AMC is the same as the group CAM, so only one of the $3!$ permutations of this group should be in the hat for selection. Thus the number of ways to select a subset of three individuals from the five friends to receive the three tickets is

$$\frac{60}{3!} = \frac{\mathbf{P}(5,3)}{3!} = \frac{5!}{2!3!}.$$

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5,3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5, 3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5,3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5,3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5, 3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5, 3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

Alan has listed all **Combinations** of the five friends, taken 3 at a time to put in the hat. The number of such combinations, which is 10, is denoted by $\mathbf{C}(5,3)$. In terms of set theory he has listed all subsets of 3 objects in the set of 5 objects $\{A, B, C, D, E\}$.

Definition A **Combination** of **n** objects taken **R** at a time is a selection (Sample, Team) of R objects taken from among the n. The order in which the objects are chosen does not matter.

The key characteristics of a combination are

1. A combination selects elements from a single set.
2. Repetitions are not allowed.
3. The order in which the selected elements are arranged is **not** significant.

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n, R)$ or $\binom{n}{R}$. We have

$$\mathbf{C}(n, R) = \binom{n}{R} = \frac{n \times (n-1) \cdot \ldots \cdot (n-R+1)}{R \cdot (R-1) \cdot (R-2) \cdots 1} = \frac{\mathbf{P}(n, R)}{R!} = \frac{n!}{R! \cdot (n-R)!}$$

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n,R)$ or $\binom{n}{R}$. We have

$$\mathbf{C}(n,R) = \binom{n}{R} = \frac{n \times (n-1) \cdot \ldots \cdot (n-R+1)}{R \cdot (R-1) \cdot (R-2) \cdots 1} = \frac{\mathbf{P}(n,R)}{R!} = \frac{n!}{R! \cdot (n-R)!}$$

Example Evaluate $\mathbf{C}(10,3)$.

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n,R)$ or $\binom{n}{R}$. We have

$$\mathbf{C}(n,R) = \binom{n}{R} = \frac{n \times (n-1) \cdot \ldots \cdot (n-R+1)}{R \cdot (R-1) \cdot (R-2) \cdots 1} = \frac{\mathbf{P}(n,R)}{R!} = \frac{n!}{R! \cdot (n-R)!}$$

Example Evaluate $\mathbf{C}(10,3)$.

$$\mathbf{C}(10,3) = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 5 \cdot 3 \cdot 8 = 120$$

Combinations

The **number of such combinations** is denoted by the symbol $\mathbf{C}(n, R)$ or $\binom{n}{R}$. We have

$$\mathbf{C}(n, R) = \binom{n}{R} = \frac{n \times (n-1) \cdot \ldots \cdot (n-R+1)}{R \cdot (R-1) \cdot (R-2) \cdots 1} = \frac{\mathbf{P}(n, R)}{R!} = \frac{n!}{R! \cdot (n-R)!}$$

Example Evaluate $\mathbf{C}(10, 3)$.

$$\mathbf{C}(10, 3) = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 5 \cdot 3 \cdot 8 = 120$$

Combinations

Example(Choosing a team) How many ways are there to choose a team of 7 people from a class of 40 students in order to make a team for Bookstore Basketball?

Combinations

Example(Choosing a team) How many ways are there to choose a team of 7 people from a class of 40 students in order to make a team for Bookstore Basketball?

$$\mathbf{C}(40,7) = \frac{40!}{7! \cdot 33!} = \frac{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$$

$$10 \cdot 39 \cdot 38 \cdot 37 \cdot 34 = 18,643,560$$

Combinations

Example(Round Robins) In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

Combinations

Example(Round Robins) In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

$$\mathbf{C}(15, 2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

Combinations

Example(Round Robins) In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

$$\mathbf{C}(15,2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

Example(Poker Hands) A poker hand consists of five cards dealt at random from a standard deck of 52. How many different poker hands are possible?

Combinations

Example(Round Robins) In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

$$\mathbf{C}(15,2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

Example(Poker Hands) A poker hand consists of five cards dealt at random from a standard deck of 52. How many different poker hands are possible?

$$\mathbf{C}(52,5) = \frac{52!}{5! \cdot 47!} = 2,598,960$$

Combinations

Example A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs. How many poker hands consist entirely of clubs?

Combinations

Example A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs. How many poker hands consist entirely of clubs?

$$\mathbf{C}(13, 5) = \frac{13!}{5! \cdot 8!} = 1,287$$

Combinations

Example A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs. How many poker hands consist entirely of clubs?

$$\mathbf{C}(13,5) = \frac{13!}{5! \cdot 8!} = 1,287$$

Example How many poker hands consist of red cards only?

Combinations

Example A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades and 13 clubs. How many poker hands consist entirely of clubs?

$$\mathbf{C}(13, 5) = \frac{13!}{5! \cdot 8!} = 1,287$$

Example How many poker hands consist of red cards only?

There are 26 red cards so $\mathbf{C}(26, 5) = \frac{26!}{5! \cdot 21!} = 65,780$

Combinations

Example How many poker hands consist of 2 kings and 3 queens?

Combinations

Example How many poker hands consist of 2 kings and 3 queens?

There are 4 kings and 4 queens. We can select 2 kings in $\mathbf{C}(4,2)$ ways and we can select 3 queens in $\mathbf{C}(4,3)$ ways. We can distinguish kings from queens so the answer is $\mathbf{C}(4,2) \cdot \mathbf{C}(4,3) = 6 \cdot 4 = 24.$

Bonus meditation: Why is

$$\mathbf{C}(8,5) = \mathbf{C}(4,4) \cdot \mathbf{C}(4,1) + \mathbf{C}(4,3) \cdot \mathbf{C}(4,2) + \mathbf{C}(4,2) \cdot \mathbf{C}(4,3) + \mathbf{C}(4,1) \cdot \mathbf{C}(4,4) \ ?$$

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

$\mathbf{C}(50, 5) = 2,118,760.$

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

$\mathbf{C}(50, 5) = 2, 118, 760.$

Example If a box of 50 light bulbs contains 20 defective light bulbs and 30 non-defective light bulbs, how many samples of size 5 can be drawn from the box so that all of the light bulbs in the sample are good?

Combinations

Example (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 and checking if any of those 5 bulbs are defective. If at least one defective bulb is found the box is not shipped, otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

$\mathbf{C}(50, 5) = 2,118,760.$

Example If a box of 50 light bulbs contains 20 defective light bulbs and 30 non-defective light bulbs, how many samples of size 5 can be drawn from the box so that all of the light bulbs in the sample are good?

$\mathbf{C}(30, 5) = 142,506.$

Problems using a mixture of counting principles

Example How many poker hands have at least two kings?

Problems using a mixture of counting principles

Example How many poker hands have at least two kings?

There are $\mathbf{C}(4,2)$ ways to get 2 kings and $\mathbf{C}(48,3)$ ways to fill out the hand. Hence there are $\mathbf{C}(4,2)\cdot\mathbf{C}(48,3)$ hands with exactly 2 kings. There are $\mathbf{C}(4,3)\cdot\mathbf{C}(48,2)$ hands with exactly 3 kings and there are $\mathbf{C}(4,4)\cdot\mathbf{C}(48,1)$ hands with exactly 4 kings. Hence there are
$\mathbf{C}(4,2)\cdot\mathbf{C}(48,3)+\mathbf{C}(4,3)\cdot\mathbf{C}(48,2)+\mathbf{C}(4,4)\cdot\mathbf{C}(48,1)$
hands with at least two kings. The number is
$6\cdot 17,296+4\cdot 1,128+1\cdot 48=108,336$.

Problems using a mixture of counting principles

Example In the Notre Dame Juggling club, there are 5 graduate students and 7 undergraduate students. All would like to attend a juggling performance in Chicago. However, they only have funding from Student Activities for 5 people to attend. The funding will only apply if at least three of those attending are undergraduates. In how many ways can 5 people be chosen to go to the performance so that the funding will be granted?

Problems using a mixture of counting principles

Example In the Notre Dame Juggling club, there are 5 graduate students and 7 undergraduate students. All would like to attend a juggling performance in Chicago. However, they only have funding from Student Activities for 5 people to attend. The funding will only apply if at least three of those attending are undergraduates. In how many ways can 5 people be chosen to go to the performance so that the funding will be granted?

Again it is useful to break the problem up, in this case by number-of-undergraduates. We need to work out how many ways we can get 3 undergraduates, how many ways we can get 4 undergraduates, how many ways we can get 5 undergraduates, and then add these numbers.

Problems using a mixture of counting principles

Again it is useful to break the problem up, in this case by number-of-undergraduates. We need to work out how many ways we can get 3 undergraduates, how many ways we can get 4 undergraduates, how many ways we can get 5 undergraduates, and then add these numbers.
Three undergraduates: $\mathbf{C}(7,3) \cdot \mathbf{C}(5,2)$; Four undergraduates: $\mathbf{C}(7,4) \cdot \mathbf{C}(5,1)$; Five undergraduates: $\mathbf{C}(7,5) \cdot \mathbf{C}(5,0)$. The number is
$35 \cdot 10 + 35 \cdot 5 + 21 \cdot 1 = 546$.

Problems using a mixture of counting principles

Again it is useful to break the problem up, in this case by number-of-undergraduates. We need to work out how many ways we can get 3 undergraduates, how many ways we can get 4 undergraduates, how many ways we can get 5 undergraduates, and then add these numbers.

Three undergraduates: $\mathbf{C}(7,3) \cdot \mathbf{C}(5,2)$; Four undergraduates: $\mathbf{C}(7,4) \cdot \mathbf{C}(5,1)$; Five undergraduates: $\mathbf{C}(7,5) \cdot \mathbf{C}(5,0)$. The number is $35 \cdot 10 + 35 \cdot 5 + 21 \cdot 1 = 546$.

Remark: $\mathbf{C}(7,3) \cdot \mathbf{C}(9,2) = 1,260$. Why is this NOT the right answer?

Problems using a mixture of counting principles

Again it is useful to break the problem up, in this case by number-of-undergraduates. We need to work out how many ways we can get 3 undergraduates, how many ways we can get 4 undergraduates, how many ways we can get 5 undergraduates, and then add these numbers.

Three undergraduates: $\mathbf{C}(7,3) \cdot \mathbf{C}(5,2)$; Four undergraduates: $\mathbf{C}(7,4) \cdot \mathbf{C}(5,1)$; Five undergraduates: $\mathbf{C}(7,5) \cdot \mathbf{C}(5,0)$. The number is $35 \cdot 10 + 35 \cdot 5 + 21 \cdot 1 = 546.$

Remark: $\mathbf{C}(7,3) \cdot \mathbf{C}(9,2) = 1,260.$ Why is this NOT the right answer? $350 + \frac{4!}{3! \cdot 1!} \cdot 175 + \frac{5!}{3! \cdot 2!} \cdot 21 = 1,260.$

Problems using a mixture of counting principles

Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

Problems using a mixture of counting principles

Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

Pat's choices are independent so
$\mathbf{C}(3,1) \cdot \mathbf{C}(2,1) \cdot \mathbf{C}(4,2) \cdot \mathbf{C}(3,2) = 3 \cdot 2 \cdot 6 \cdot 3 = 108.$

Problems using a mixture of counting principles

Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

Pat's choices are independent so
$\mathbf{C}(3,1) \cdot \mathbf{C}(2,1) \cdot \mathbf{C}(4,2) \cdot \mathbf{C}(3,2) = 3 \cdot 2 \cdot 6 \cdot 3 = 108.$

Example How many subsets of a set of size 5 have at least 4 elements?

Problems using a mixture of counting principles

Example Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 vegetable toppings and 3 meat toppings. Pat always chooses one type of crust, one type of cheese, 2 vegetable toppings and two meat toppings. How many different pizzas can Pat create?

Pat's choices are independent so
$\mathbf{C}(3,1)\cdot\mathbf{C}(2,1)\cdot\mathbf{C}(4,2)\cdot\mathbf{C}(3,2) = 3\cdot 2\cdot 6\cdot 3 = 108.$

Example How many subsets of a set of size 5 have at least 4 elements?

$\mathbf{C}(5,4)+\mathbf{C}(5,5).$

Special Cases and Formulas

- It is immediate from our formula

$$\mathbf{C}(n,k) = \frac{n!}{k!(n-k)!}$$

  that $$\mathbf{C}(n,k) = \mathbf{C}(n,n-k).$$

- $\mathbf{C}(n,0) = 1$, which makes sense since there is exactly one subset with zero elements in every set. For our formula to always hold we define $0! = 1$.
- $\mathbf{C}(n,1) = \frac{n!}{(n-1)!} = n$, which makes sense since there are exactly n subsets with one element in a set with n elements.
- $\mathbf{C}(n,n) = \mathbf{C}(n,0) = 1$ which makes sense because .... (finish the sentence).

The Binomial Theorem

The Binomial theorem says that for any positive integer n and two numbers x and y, we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

For example if n = 4, then the theorem says that

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Example (a) Check that the above equation is true for $n = 4, x = 2, y = 3.$

The Binomial Theorem

The Binomial theorem says that for any positive integer n and two numbers x and y, we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

For example if n = 4, then the theorem says that

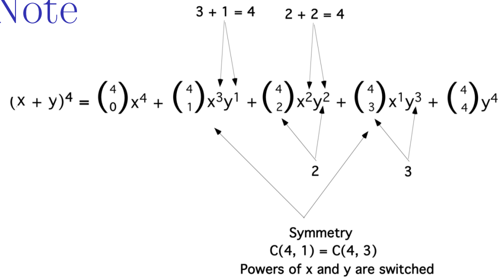
$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Example (a) Check that the above equation is true for $n = 4, x = 2, y = 3$.

$(2+3)^4 = \binom{4}{0}2^4 + \binom{4}{1}2^3 \cdot 3 + \binom{4}{2}2^2 \cdot 3^2 + \binom{4}{3}2 \cdot 3^3 + \binom{4}{4}3^4$.

(b) Check that the above equation is true for $n = 4, x = 1, y = 1$.

Things to Note



Note how the powers relate to the lower numbers in the coefficients.

Note how the powers decrease on the x's and increase on the y's as we move from left to right.

Note how the powers in each **term** of the expression add to n.

Note the symmetry in the expansion.

$\binom{4}{1}x^3y$ is a **Term** of the expression and $\binom{4}{1}$ is the coefficient of that term.

Things to Note

Note that the sum of the coefficients of the terms of the expansion of $(x+y)^4$ is equal to the total number of subsets one can make using a set of size $n = 4$. Setting $x = 1$ and $y = 1$ in the above equation, we get the formula:

$$2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

Applying the same proof to the general case, we get the formula:

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$2^n = \mathbf{C}(n,0) + \mathbf{C}(n,1) + \mathbf{C}(n,2) + \mathbf{C}(n,3) + \cdots + \mathbf{C}(n,n)$$

for any counting number n. Thus **the number of subsets of a set with n elements is 2^n.**

Things to Note

Example A set has ten elements. How many of its subsets have at least two elements?

Things to Note

Example A set has ten elements. How many of its subsets have at least two elements?

$C(10,2) + \mathbf{C}(10,3) + \mathbf{C}(10,4) + \mathbf{C}(10,5) + \mathbf{C}(10,6) + \mathbf{C}(10,7) + \mathbf{C}(10,8) + \mathbf{C}(10,9) + \mathbf{C}(10,10).$

To actually compute this number it is easier to compute

$$2^{10} - \big(\mathbf{C}(10,0) + \mathbf{C}(10,1)\big) = 1024 - (1 + 10) = 1013$$

Example How many tips could you leave at a restaurant, if you have a half-dollar, a one dollar coin, a two dollar note and a five dollar note?

Things to Note

Example A set has ten elements. How many of its subsets have at least two elements?

$C(10,2) + \mathbf{C}(10,3) + \mathbf{C}(10,4) + \mathbf{C}(10,5) + \mathbf{C}(10,6) + \mathbf{C}(10,7) + \mathbf{C}(10,8) + \mathbf{C}(10,9) + \mathbf{C}(10,10).$

To actually compute this number it is easier to compute

$$2^{10} - \big(\mathbf{C}(10,0) + \mathbf{C}(10,1)\big) = 1024 - (1 + 10) = 1013$$

Example How many tips could you leave at a restaurant, if you have a half-dollar, a one dollar coin, a two dollar note and a five dollar note?

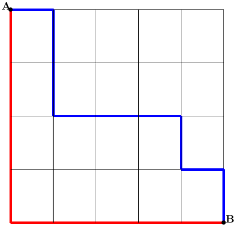
You can leave any subset of your money. You have 4 items so there are $2^4 = 16$ possibilities.

Taxi Cab Geometry revisited

Recall that the number of taxi cab routes (always traveling south or east) from A to B is the number of different rearrangements of the sequence SSSSEEEEE which is

$$\frac{9!}{4!5!} = \mathbf{C}(9,4) = \mathbf{C}(9,5).$$

The sequence SSSSEEEEE is shown in red and the sequence ESSEEEESES in blue.



Can you think of any reason why the number of routes should equal the number of ways to choose 4 objects from a set of 9 objects?

Taxi Cab Geometry revisited

Can you think of any reason why the number of routes should equal the number of ways to choose 4 objects from a set of 9 objects?

A route is specified by a string of S's and E's with exactly 4 S's and 5 E's and any such string specifies a route. In other words you need a nine letter word made up of 4 S's and 5 E's. If I give you a subset of 4 elements of the set $\{1, 2, \cdots, 9\}$ and you put S's in those positions, then you can fill in the rest of the word with E's.

As an example, given $\{3, 1, 7, 6\}$ start with
S_S_ _SS_ _ and get S E S E E S S E E.