## A formula for the optimal strategy for $R$ and $C$

and the value of the game for $2 \times 2$ payoff matrices.
In the case of $2 \times 2$ payoff matrices with no saddle point, we can derive a formula for the optimal strategies for both players.
Definition An equilibrium point of a game where both players may use mixed strategies is a pair of mixed strategies such that neither player has any incentive to unilaterally change to another mixed strategy.
When searching for optimal mixed strategies for both players, we assume a number of things:

- The pay-off matrix is known to both players.
- Whatever mixed strategy is played by either player can be deduced by the opponent by observation.
- Both players strive to maximize their expected payoff (note that in a zero sum game, the expected payoffs of the players add to zero, therefore maximizing an expected payoff for one player is equivalent to minimizing the expected payoff of the other player).
- Whatever mixed strategy a player chooses, their opponent will choose the best counterstrategy.

In view of these assumptions, we see that if optimal mixed strategies exist for both players these strategies should occur at an equilibrium point, since neither player should be able to increase their payoff by unilaterally changing their strategy. The Minimax theorem below shows that optimal mixed strategies for both players always exist for a zero sum game.
Minimax Theorem: John Von Neumann For every zero sum game, there is a number $\nu$ (for value) and particular mixed strategies (optimal mixed strategies) for both players such that

1. The expected payoff to the row player will be at least $\nu$ if the row player plays his or her particular mixed strategy, no matter what mixed strategy the column player plays.
2. The expected payoff to the row player will be at most $\nu$ if the column player plays his or her particular mixed strategy, no matter what strategy the row player chooses.
the number $\nu$ is called the value of the game and represents the expected advantage to the row player (a disadvantage if $\nu$ is negative).

## Calculating Optimal mixed strategies and the value of the game for a $2 \times 2$ payoff matrix with no Saddle Point algebraically

We demonstrate the general principle with an example here.
Example Consider the situation where the payoff matrix for the row player is given by

$$
\left[\begin{array}{ll}
7 & 3 \\
2 & 5
\end{array}\right]
$$

We denote the row player by R and the column player by C . We denote R 's mixed strategy by $[p, 1-p]$ for some value of $p$ with $0 \leq p \leq 1$ and C's mixed strategy by $\left[\begin{array}{c}q \\ 1-q\end{array}\right]$ for some value of $q$ with $0 \leq q \leq 1$. We wish to determine the values of $p$ and $q$ which give optimal strategies for both players.

We know that the expected vale of R's payoff is given by

$$
\begin{gathered}
{\left[\begin{array}{ll}
p & 1-p
\end{array}\right]\left[\begin{array}{ll}
7 & 3 \\
2 & 5
\end{array}\right]\left[\begin{array}{c}
q \\
1-q
\end{array}\right]=\left[\begin{array}{ll}
7 p+2(1-p) & 3 p+5(1-p)
\end{array}\right]\left[\begin{array}{c}
q \\
1-q
\end{array}\right]} \\
=\left[\begin{array}{cc}
5 p+2 & 5-2 p
\end{array}\right]\left[\begin{array}{c}
q \\
1-q
\end{array}\right]=(5 p+2) q+(5-2 p)(1-q)=5 p q+2 q+5-2 p-5 q+2 p q \\
=5+7 p q-3 q-2 p=29 / 7+7(p-3 / 7)(q-2 / 7)
\end{gathered}
$$

If R sets $p$ equal to $3 / 7$, then R is guaranteed an expected payoff of $29 / 7$. If $R$ chooses any other value of $p$, C can choose a value of $q$ that will make the expression $7(p-3 / 7)(q-2 / 7)$ negative giving a payoff for R which is less than $29 / 7$.
(If R sets $p<3 / 7$ then the expression $7(p-3 / 7)(q-2 / 7)$ is negative as long as $q>2 / 7$, if $p>3 / 7$ then the expression $7(p-3 / 7)(q-2 / 7)$ is negative as long as $q<2 / 7$. )

Thus an expected payoff of $29 / 7$ is the best that R can do if C uses an optimal counterstrategy. Thus R's optimal mixed strategy is $(3 / 7,4 / 7)$.

Similarly C's best counterstrategy is to set $q$ equal to $2 / 7$ since otherwise R can choose a value of $p$ making the expression $7(p-3 / 7)(q-2 / 7)$ positive to give an expected payoff for R that is greater than $29 / 7$ (thus a payoff for C which is less than $-29 / 7$ ).
Thus C's optimal strategy is given by $\left[\begin{array}{l}2 / 7 \\ 5 / 7\end{array}\right]$

Since the expected pay-off is given by $29 / 7+7(p-3 / 7)(q-2 / 7)$,
the value of the game is given by $\nu=29 / 7$ (when both players play their optimal strategy).

## Optimal Mixed Strategies and the Value of a Zero-Sum Game with no saddle point

(If there is a saddle point in the matrix, we saw before that the best strategy is a fixed strategy for both players. The reasoning above breaks down in this case because either the derived value of $p$ or $q$ is not in the interval $[0,1]$ or $a+d-b-c=0$.)

A similar calculation to that given above with the general payoff matrix for the row player given by

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

( where $a, b, c$ and $d$ are not all the same i.e. $a-b-c+d \neq 0$ ) yields an expected payoff for strategies $\left[\begin{array}{cc}p & 1-p\end{array}\right]$ and $\left[\begin{array}{c}q \\ 1-q\end{array}\right]$ by the row and column players respectively of

$$
\frac{a d-b c}{a-b-c+d}-(a-b-c+d)\left(p-\frac{d-c}{a-b-c+d}\right)\left(q-\frac{d-b}{a-b-c+d}\right) .
$$

As above, we deduce that

- The optimal strategy for the row player is to set the probability of playing Row 1 equal to

$$
p=\frac{d-c}{a-b-c+d} .
$$

The row player's probability of playing Row 2 is then determined as $1-p$.

- The optimal strategy for the column player is to set the probability of playing Column 1 equal to

$$
q=\frac{d-b}{a-b-c+d}
$$

The column player's probability of playing Column 2 is then determined as $1-q$.

- The value of the game (expected payoff for the row player if both players play optimally) is given by

$$
\nu=\frac{a d-b c}{a-b-c+d} .
$$

The expected payoff for the column player is given by the negative of the expected payoff for the row player since it is a zero sum game.

- The game is called a fair game if the value of the game is $\nu=0$.

This method can be used for any payoff matrix which is $2 \times 2$ or for and payoff matrix that has a reduced matrix which is $2 \times 2$. It can also be used for constant sum games.

Example The following represents the payoff matrix for a two person zero-sum game with row player $R$ and column player $C$;

$$
\left[\begin{array}{cc}
5 & -1 \\
2 & 4
\end{array}\right]
$$

(a) Is this a strictly determined game?
(b) What are the optimal strategies for R and C for this game?
(c) What is the value of the game.
(d) Is this a fair game?

Example Example: Football Run or Pass? [Winston] In football, the offense selects a play and the defense lines up in a defensive formation. We will consider a very simple model of play selection in which the offense and defense simultaneously select their play. The offense may choose to run or to pass and the defense may choose a run or a pass defense. One can use the average yardage gained or lost in this particular League as payoffs and construct a payoff matrix for this two player zero-sum game. In a previous lecture, we constructed a payoff matrix for this example based on the given statistics for the two teams involved.

|  |  | Defense |  |
| :--- | :--- | :---: | :---: |
|  |  | Run <br> Defense | Pass <br> Defense |
| Offense | Run | -5 | 5 |
|  | Pass | 10 | 0 |

(a) What is the optimal strategy for both teams in this game?
(b) What is the value of the game given that both teams play optimally?
(c) What do these numbers mean?

## Dominant Rows and columns

In a payoff matrix, we say Row $i$ dominates Row $j$ if every entry of Row $i$ is greater than or equal to the corresponding entry of Row $j$.

We say that Column $i$ dominates Column $j$ if every entry in Column $i$ is less than or equal to the corresponding entry in Column $j$.

Whenever Row $i$ dominates Row $j$, Row $j$ can be removed from the payoff matrix without affecting the analysis of optimal strategies and game value (because the row player would have no reason to ever play Row $j$ ).

Whenever Column $i$ dominates Column $j$, Column $j$ can be removed from the payoff matrix without affecting the analysis of optimal strategies and game value (because the column player would have no reason to ever play Column $j$ ).

Reduced Payoff matrix When we have removed all dominated rows and columns from a matrix, new dominated rows and columns may appear in the reduced matrix. We can also remove these rows and columns from the new matrix without affecting the game analysis and continue in the same fashion until we arrive at a matrix with no dominated rows or columns. This matrix is called the reduced payoff matrix. If the game has a unique saddle point, the reduced matrix will be a $1 \times 1$ matrix whose unique entry is the value of the game. (As we reduce a matrix, we should keep track of the original names of the row and column strategies to determine the best strategy). Obviously if we can reduce a payoff matrix to a $2 \times 2$ matrix, we can determine the optimal strategy.

Example Determine the optimal strategy and the resulting value for this mixed strategy game

$$
\left[\begin{array}{lll}
6 & 7 & 3 \\
4 & 5 & 2 \\
5 & 6 & 8
\end{array}\right]
$$

Example A baseball pitcher throws three pitches, a fastball, a slider and a change-up. As a measure of the payoff for this type of confrontation, we might use the expected number of runs the batter creates in each situation. (Note their are other possible measures that take into account the pitcher's abilities). We would expect that for any given pitch, the batter's performance is better if he anticipates the pitch. Lets assume that the batter has four possible strategies, To anticipate either a fastball, a slider or a change-up or not to anticipate any pitch.

|  | Pitcher |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | Fastball |  |  |  |
| Batter | Change-up | Slider |  |  |
|  | Fastball | 0.38 | 0.37 | 0.39 |
|  | Slider | 0.35 | 0.32 | 0.45 |
|  | None | 0.38 | 0.3 | 0.41 |

(a) Find the reduced payoff matrix.
(b) Find the optimal mixed strategy for both players.
(c) Find the value of the game.
(d) Interpret the value of the game.

## Sample Exam Questions

1 Ross and Carter play a game where the payoff matrix for Ross is given by:

$$
\left[\begin{array}{cc}
-2 & 1 \\
2 & -1
\end{array}\right]
$$

Which of the following gives the optimal strategy for Carter?
(a) $\left[\begin{array}{l}1 / 3 \\ 2 / 3\end{array}\right]$
(b) $\left[\begin{array}{l}2 / 3 \\ 1 / 3\end{array}\right]$
(c) $\left[\begin{array}{l}1 / 6 \\ 5 / 6\end{array}\right]$
(d) $\left[\begin{array}{l}1 / 2 \\ 1 / 2\end{array}\right]$
(e) $\left[\begin{array}{l}0 \\ 1\end{array}\right]$

2 Ratman and Catman play a game where the payoff matrix for Ratman is given by:

$$
\left[\begin{array}{cc}
-1 & 1 \\
3 & -3
\end{array}\right]
$$

Which of the following is true?
(a) This is a strictly determined game with saddle point at Row 2, Column 1
(b) This is a fair game
(c) The optimal strategy for R is a pure strategy of $[1,0]$.
(d) The optimal strategy for $C$ is a pure strategy of $\left[\begin{array}{l}0 \\ 1\end{array}\right]$.
(e) None of the above.

3 Roger and Connor play a game where the payoff matrix for Roger is given by:

$$
\left[\begin{array}{ccc}
1 & 4 & 2 \\
2 & 5 & 1 \\
-2 & 6 & -1
\end{array}\right]
$$

Which of the following gives the optimal strategy for Roger?
(a) $\left[\begin{array}{lll}1 / 2 & 1 / 2 & 0\end{array}\right]$
(b) $\left[\begin{array}{lll}1 / 3 & 1 / 3 & 1 / 3\end{array}\right]$
(c) $\left[\begin{array}{lll}1 / 2 & 0 & 1 / 2\end{array}\right]$
(d) $\left[\begin{array}{lll}1 & 0 & 0\end{array}\right]$
(e) $\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]$

