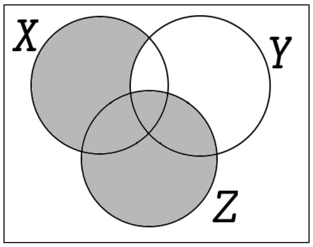
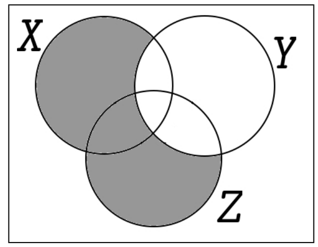
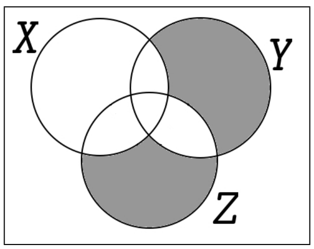
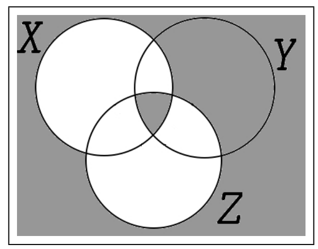
Multiple Choice

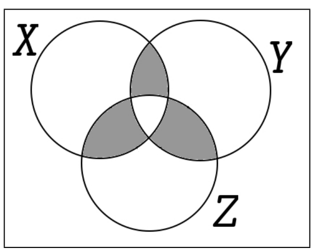
1.(5pts) Which Venn diagram below has $(X \cap Y^c) \cup Z$ shaded?

(a) 

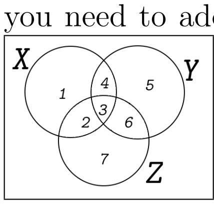
(b) 

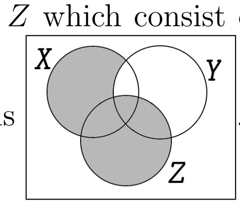
(c) 

(d) 

(e) 

Solution. You need to shade $X \cap Y^c$ which is the set of things in X which are not in Y. Hence you need to shade at least the regions numbered 1 and 2 in the picture below. Then you need to add all the regions in Z which consist of the regions numbered 2, 3, 6 and 7.



Hence the answer is 

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2.(5pts) If $A = \{1, 3, 5, \cdots, 11\}$, $B = \{1, 4, 7, 10, 13\}$ and $D = \{2, 4, 6, \cdots, 20\}$ find $(A \cup B) \cap D$.

(a) $\{4, 10\}$ (b) $\{1, 7, 10, 13\}$ (c) $\{12, 14, 16, 18, 20\}$

(d) $\{3, 5, 9, 11\}$ (e) $\emptyset$, the empty set

Solution. $A \cup B = \{1, 3, 4, 5, 7, 9, 10, 11, 13\}$ and $\{1, 3, 4, 5, 7, 9, 10, 11, 13\} \cap \{2, 4, 6, \cdots, 20\} = \{4, 10\}$.

3.(5pts) A deli offers 5 different types of bread, 4 types of meat and 6 types of vegetables. A sandwich must have one bread and at least one meat or one vegetable. It can have up to all 4 meats and all 6 vegetables. How many sandwich options does this deli offer?

(a) $5,115$ (b) $4,415$ (c) $1,275$ (d) $1,530$ (e) $32,736$

Solution. I have 5 choices for the bread. For my meat choices I need some subset of all of them so there are $2^4 = 16$ choices of meat (including the empty set which means no meat). Likewise there are $2^6 = 64$ choices of vegetables. Hence I have $16 \cdot 64 = 1,024$ choices of meat and vegetables. *However* I cannot have no meat and no vegetables so I only have $1,023$ choices of filling so the total number of choices is $5 \cdot 1,023 = 5,115$.

4.(5pts) Let

$$U = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

be the Latin alphabet. Let B be the set of letters in the word "baboon". Find B'. In all the answers, the letters are in alphabetical order.

(a) $\{c, d, e, f, g, h, i, j, k, l, m, p, q, r, s, t, u, v, w, x, y, z\}$

(b) $\{c, d, e, f, g, i, j, k, l, m, p, q, r, s, t, u, v, w, x, y, z\}$

(c) $\{c, d, e, f, h, i, j, k, l, m, p, q, r, s, t, u, w, x, y, z\}$

(d) $\{c, d, e, f, g, h, i, j, k, l, m, p, q, s, t, u, w, x, y, z\}$

(e) $\{a, d, e, f, g, h, i, j, k, l, m, p, q, r, s, t, u, v, w, x, y, z\}$

Solution. The answer is the the alphabet with the red letters removed.
$\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$

5.(5pts) You need to drive from here to Chicago. Alas your car is not very reliable so you plan to make the trip in three segments. There are 3 routes from here to Michigan City. Then there are 5 routes you can take to Gary. Once in Gary there are 3 routes into Chicago. How many routes are possible.

(a) 45 (b) 11 (c) 15 (d) 22 (e) $1,671$

Solution. You have 3 choices for the first leg of your trip. Then you have 5 choices for the next leg and 3 choices for the last leg, so the answer is $3 \cdot 5 \cdot 3 = 45$.

6.(5pts) Tara has 20 books and is allowed to bring at most three on vacation. How many subsets of Tara's twenty books have at most three elements?
Note: no books is an option.

(a) $1,351$ (b) 211 (c) $1,140$

(d) $1,048,576$ (e) 56

Solution. We want to count the number of subsets with either 0 elements, 1 element, 2 elements or three elements. Since the set of subsets with k elements is disjoint from the set of subsets with r elements if $k \neq r$, we can use the addition principle.

- The number of subsets with 0 elements of a set of 20 elements is $C(20,0) = 1$.
- The number of subsets with 1 element of a set of 20 elements is $C(20,1) = 20$.
- The number of subsets with 2 elements of a set of 20 elements is $C(20,2) = \frac{20\cdot 19}{2\cdot 1} = 190$.
- The number of subsets with 3 elements of a set of 20 elements is $C(20,3) = \frac{20\cdot 19\cdot 18}{3\cdot 2\cdot 1} = 1,140$

Therefore the number of subsets of a set with 20 elements which have at most three elements is $C(20,0) + C(20,1) + C(20,2) + C(20,3) = 1 + 20 + 190 + 1,140 = 1,251$.

7.(5pts) When ordering a burger at Paul's Famous Hamburger's in Sydney, you must first choose one type of meat, from Chicken or Beef. You then choose a subset of the eight optional fillings, tomato, lettuce, egg, bacon, cheese, pineapple, beetroot and cooked onions for your burger. After you have chosen your preferred subset of fillings, you choose one sauce from the five available sauces, BBQ, Sweet Chili, Hot Chili, Mustard and Paul's special sauce. If you wish to order a burger with at most two fillings, how many different burgers are possible?
Note: No fillings is a possibility.

(a) 370 (b) 280 (c) $2,560$ (d) $2,470$ (e) 560

Solution. We apply a mixture of counting techniques here. First we break the task of ordering the hamburger into steps and use the multiplication principle to count the number of ways of completing the task.

- Step 1: choose a type of meat $\rightarrow$ 2 ways.
- Step 2: choose a subset of at most two (either 0, 1 or 2) fillings from the eight available $\rightarrow C(8,0) + C(8,1) + C(8,2)$ ways.
- Step 3: choose a sauce $\rightarrow$ 5 ways.

(In step 2, we count the number of subsets of a set of size 8, with either 0 elements, 1 elements or 2 elements. Since the set of subsets with k elements is disjoint from the set of subsets with r elements if $k \neq r$, we can use the addition principle.)

Using the multiplication principle, we get that the number of different burgers we can order with at most two fillings is

$$2 \cdot (C(8,0) + C(8,1) + C(8,2)) \cdot 5 = 2 \cdot 37 \cdot 5 = 370$$

8.(5pts) A poker hand consists of a selection of 5 cards from a standard deck of 52 cards. There are 13 denominations, aces, kings, queens, ..., twos, and four suits, hearts, diamonds, spades and clubs in a standard deck. How many poker hands have three aces and two cards which are not aces but which are of the same denomination?

(a) 288 (b) 24 (c) 6 (d) 48 (e) 156

Solution. We can break the task of constructing such a hand into steps and use the multiplication principle to count the number of such hands.

- Step 1: Choose 3 Aces $\rightarrow C(4,3) = 4$ ways.
- Step 2: Choose a different denomination $\rightarrow 12$ ways.
- Step 3: Choose 2 cards from the new denomination $\rightarrow C(4,2) = 6$ ways

The total number of such hands is $4 \cdot 12 \cdot 6 = 288$

9.(5pts) A class of 15 students is visiting the Louvre and the teacher wants to take a photograph of 5 of them lined up under the Mona Lisa. How many such photographs are possible?

(a) $P(15,5)$ (b) 2^{15} (c) $15!$ (d) $C(15,5)$ (e) 15^5

Solution. The number of such photographs is

$$P(15,5) = 360,360.$$

10.(5pts) In an experiment a coin will be flipped 10 times and the resulting ordered sequence of heads and tails will be recorded. How many of the possible sequences that might result from this experiment have exactly 4 heads?

(a) 210 (b) 5,040 (c) 151,200 (d) 1,024 (e) 52,920

Solution. The number of sequences with exactly 4 H's is equal to the number of words we can make by rearranging the letters of the word HHHHTTTTTT. This is

$$\frac{10!}{4! \cdot 5!} = C(10,4) = 210$$

Partial Credit

You must show your work on the partial credit problems to receive credit!

Where applicable, answers may be given in the form of products of numbers and symbols for factorials and numbers of permutations and combinations.

11.(12pts) On your floor of your dorm there are 20 students, 5 are seniors, 5 juniors, 5 sophomores and 5 freshmen. The rector wants to meet with all of you in groups of 4.

(a) How many ways can the 5 groups be selected if does not matter which group is first, second, ...?

(b) How many ways can the 5 groups be selected if does not matter which group is first, second, ... but there is to be one member of each class in each group?

Solution. (a) The first group of four can be picked in $C(20,4)$ ways. The second group of four can be picked in $C(16,4)$ ways. The third group of four can be picked in $C(12,4)$ ways. The fourth group of four can be picked in $C(8,4)$ ways. The fifth group of four can be picked in $C(4,4)$ ways. Since the order does not matter the answer is $\dfrac{C(20,4)\cdot C(16,4)\cdot C(12,4)\cdot C(8,4)\cdot C(4,4)}{P(5,5)}$

(b) Now the first group of four can be picked in $5\cdot 5\cdot 5\cdot 5 = 5^4$ ways. The second group of four can be picked in $4\cdot 4\cdot 4\cdot 4 = 4^4$ ways. The third group of four can be picked in 3^4 ways. The fourth group of four can be picked in 2^4 ways. The fifth group of four can be picked in 1^4 ways. Since the order does not matter the answer is $\dfrac{5^4\cdot 4^4\cdot 3^4\cdot 2^4\cdot 1^4}{5!}$

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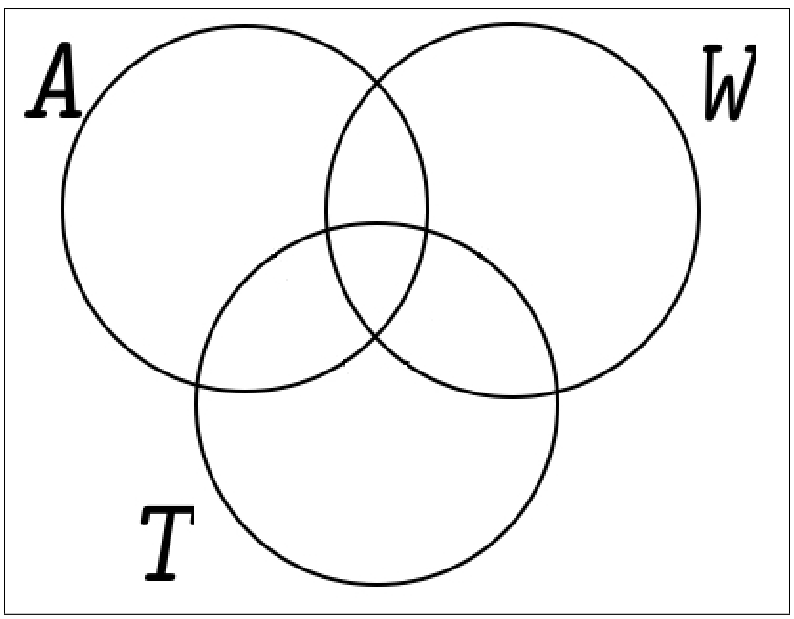
12. (12pts) Suppose given three sets A, T and W in a universe consisting of 90 elements. Suppose $n(A) = 27$, $n(T) = 28$ and $n(W) = 22$. Further suppose $n(A\cup T) = 40$, $n(T\cup W) = 38$ and $n(A\cup W) = 36$ and $n(A\cap T\cap W) = 10$. Fill in the Venn diagram below. Please work out the answer on the back of the previous page first and then neatly label the supplied diagram.

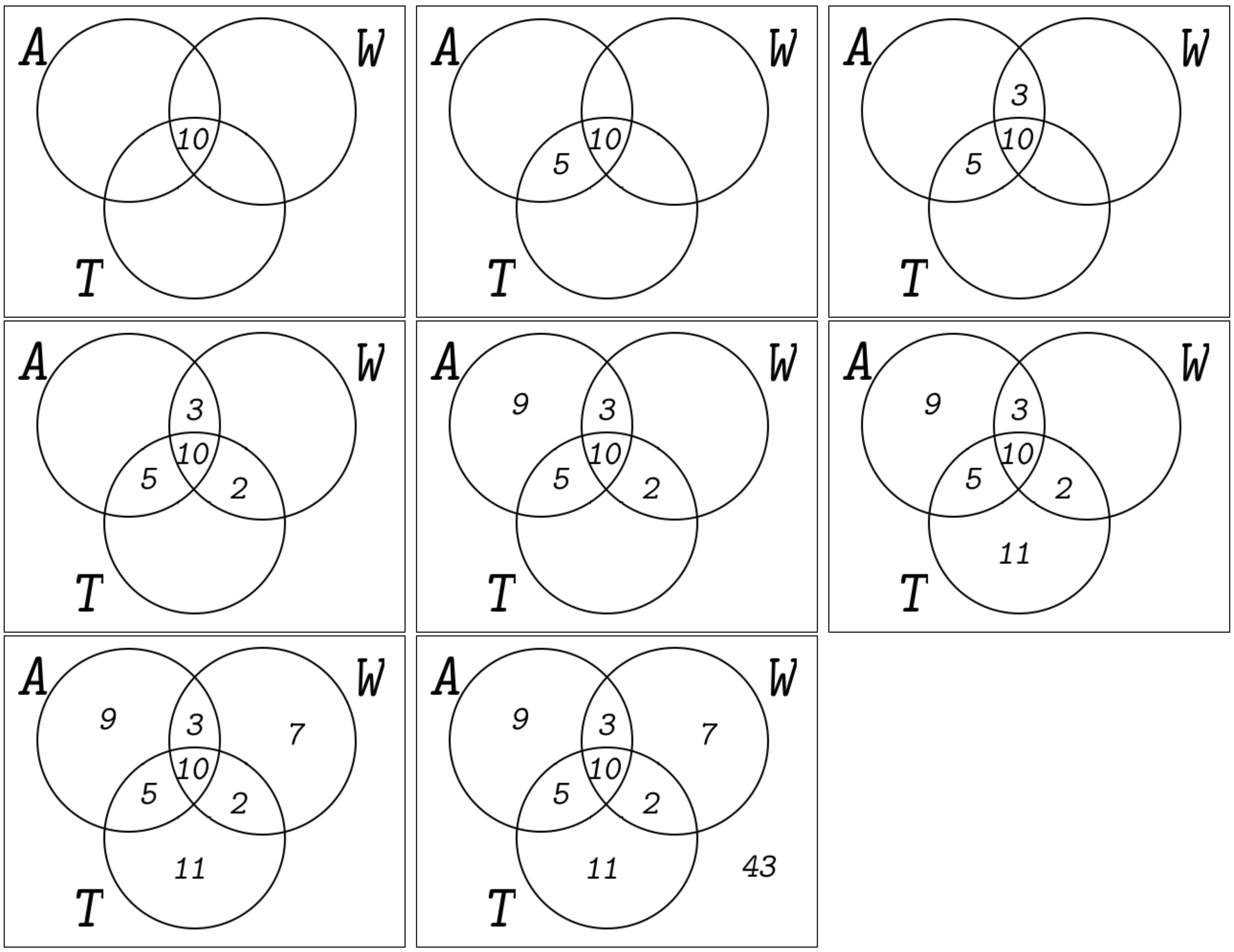
Hint: You will need the Inclusion-Exclusion Principle.



Solution. The Inclusion-Exclusion Principle is used as follows.

- $n(A \cup T) = n(A) + n(T) - n(A \cap T)$ so $40 = 27 + 28 - n(A \cap T)$ so $n(A \cap T) = 15$.
- $n(A \cup W) = n(A) + n(W) - n(A \cap W)$ so $36 = 27 + 22 - n(A \cap W)$ so $n(A \cap W) = 13$.
- $n(T \cup W) = n(T) + n(W) - n(T \cap W)$ so $38 = 28 + 22 - n(T \cap W)$ so $n(T \cap W) = 12$.

Then



13.(12pts)

SENSELESS

(a) How many words (including nonsense words) can be made by rearranging the letters of the above word?

(b) How many different 3 letter words (including nonsense words) can be made from the letters of the above word if letters can be repeated?

(c) How many different 3 letter words (including nonsense words) can be made from the letters of the above word if letters cannot be repeated?

(d) How many different 3 letter words (including nonsense words) can be made from the letters of the above word if letters cannot be repeated and the words must end with a vowel?

Solution.

(a) We are counting the number of permutations of 9 letters some of which are the same. We have 4 S's, 3 E's, 1 N and 1 L. Thus the number of different words we can form is

$$\frac{9!}{4! \cdot 3! \cdot 1! \cdot 1!} = 504$$

(b) The set of letters that we can use is $\{S, E, N, L\}$. We use the multiplication principle to count the number of ways of constructing a three letter word from the letters. We have 4 ways to choose the first letter, 4 ways to choose the second letter and 4 ways to choose the third. In all we can construct 4^3 words.

(c) The set of letters that we can use is $\{S, E, N, L\}$. We use the multiplication principle to count the number of ways of constructing a three letter word from the letters. We have 4 ways to choose the first letter, 3 ways to choose the second letter and 2 ways to choose the third. In all we can construct $4 \cdot 3 \cdot 2$ words.

(d) The set of letters that we can use is $\{S, E, N, L\}$. We use the multiplication principle to count the number of ways of constructing a three letter word from the letters. We choose the letter at the end of the word first. We have 1 way to choose the letter at the end, 3 ways to choose the middle letter and 2 ways to choose the letter at the left. In all we can construct $2 \cdot 3 \cdot 1 = 6$ words.

14. (12pts) An urn has 15 red marbles (numbered 1 through 15), 10 blue marbles (numbered 16 through 25), and 6 green marbles numbers (26 through 31) in it.
When counting the number of samples, the order of the elements in the sample is irrelevant.

(a) If you choose a sample (subset) of 5 marbles from the urn, how many different samples of size 5 are possible?

(b) How many samples of size 5 have precisely 2 red marbles, 1 blue marble, and 2 green marbles?

(c) How many samples of size 5 have 0 blue marbles and at least 3 green marbles?

Solution.

(a) The number of different samples of size 5 is $C(31, 5) = 169,911$

(b) We use the multiplication principle. In order to construct such a sample, we:
- first choose 2 red marbles $\to C(15, 2) = 105$ ways,
- then we choose 1 blue marble $\to C(10, 1) = 10$ ways,
- then we choose 2 green marbles $\to C(6, 2) = 15$ ways.

Thus the number of such samples we can make is $105 \cdot 10 \cdot 15 = 15,750$.

(c) How many samples of size 5 have 0 blue marbles and at least 3 green marbles? All of these samples will be drawn from the red and green marbles only. They will have exactly 3 green marbles(and 2 red) or exactly 4 green(and 1 red) marbles or exactly 5 green marbles. Since these samples belong in disjoint sets, we will count the number in each and add.
- The number of samples of size 5 with exactly 3 green and 2 red marbles is $C(6, 3) \cdot C(15, 2) = 20 \cdot 105 = 2,100$
- The number of samples of size 5 with exactly 4 green and 1 red marbles is $C(6, 4) \cdot C(15, 1) = 15 \cdot 15 = 225$
- The number of samples of size 5 with exactly 5 green marbles is $C(6, 5) = 6$

Thus the number of samples of size 5 with no blue blue marbles and at least 3 green marbles is

$$(C(6, 3) \cdot C(15, 2)) + (C(6, 4) \cdot C(15, 1)) + C(6, 5) = 2,100 + 225 + 6 = 2,331$$

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15. (2pts) You will get this 2 points if your instructor can read your name easily on the front page of the exam and you mark the answer boxes with an X (as opposed to a circle or any other mark).
