Section 6.7: Partitions

if we wish to divide a set of size n into disjoint subsets, there are many ways to do this.

Example Six friends Alan, Cassie, Maggie, Seth, Roger and Beth have volunteered to help at a fundraising show. One of them will hand out programs at the door, two will run a refreshments stand and three will help guests find their seats. In assigning the friends to their duties, we need to divide or partition the set of 6 friends into disjoint subsets of 3, 2 and 1. There are a number of different ways to do this, a few of which are listed below:

Prog.	Refr.	Usher
A	CM	SRB
C	AS	MRB
M	CB	ASR
B	SR	ACM
R	CM	SAB

This is not a complete list, it is not difficult to think of other possible partitions. However, we know from experience that it is easier to count the number of such partitions by using our counting principles than it is by listing all of them. We can solve this problem easily by breaking the task of assigning the friends into the three groups into steps;

Step 1: choose three of the friends to serve as ushers ($C(6,3) = \frac{6!}{3!3!}$ ways)

Step 1: choose two of the remaining friends to run the refreshment stand ($C(3,2) = \frac{3!}{2!1!}$ ways)

Step 3: choose one of the remaining friends to hand out programs ($C(1,1) = \frac{1!}{1!0!}$ ways)

Thus using the multiplication principle, we find that the number of ways that we can split the group of 6 friends into sets of 3, 2 and 1 is

$$C(6,3) \cdot C(3,2) \cdot C(1,1) = \frac{6!}{3!3!} \cdot \frac{3!}{2!1!} \cdot \frac{1!}{1!0!} = \frac{6!}{3!\cancel{3!}} \cdot \frac{\cancel{3!}}{2!\cancel{1!}} \cdot \frac{\cancel{1!}}{1!0!} = \frac{6!}{3!2!1!}$$

since $0! = 1$.

A similar calculation yields a general formula for the number of ordered partitions of a particular type of a set with n elements.

A set S is **partitioned** into k nonempty subsets $A_1, A_2, \ldots, A_n$ if:

1. Every pair of subsets in disjoint: that is $A_i \cap A_j = \emptyset$ if $i \neq j$.
2. $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Ordered partitions

A partition is **ordered** if different subset of the partition have characteristics that distinguishes one from the other.

Example In the above example, all three subsets of the partition have different sizes, so they are distinguishable from each other.

Example A If we wish to partition the group of six friends into three groups of two, and assign two to hand out programs, two to the refreshments stand and two as ushers, we have an ordered partition because the groups have different assignments. The following two partitions are counted as different ordered partitions:

Prog.	Refr.	Usher
AS	CM	RB
CM	AS	RB

(We will look at unordered partitions below where no distinction is made between the above two partitions and the order of the groups does not matter).

Formula for the number of ordered partitions

As we did in the example above, we can derive a formula for the number of ordered partitions of a set using the multiplication principle and some cancellation. We introduce a special symbol for the number below.

A set with n elements can be partitioned into k ordered subsets of $r_1, r_2, \ldots, r_k$ elements (where $r_1 + r_2 + \cdots + r_k = n$) in the following number of ways:

$$\boxed{\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!r_2!\ldots r_k!}}$$

Note If we are just partitioning a set with n elements into two sets with r_1 and r_2 elements respectively, where $r_1 + r_2 = n$, then $r_2 = n - r_1$ and the number of unordered partitions is

$$\binom{n}{r_1, n - r_1} = \frac{n}{r_1!(n - r_1)!} = C(n, r_1).$$

Example A In how many ways can the group of six friends Alan, Cassie, Maggie, Seth, Roger and Beth, be assigned to three groups of two where two are assigned to hand out programs, two are assigned to the refreshments stand and two are assigned as ushers?

Example In how many ways can a set of ten people be divided into groups of five, three and two?

Example Evaluate

$$\binom{7}{3, 2, 2}.$$

Example A group of 12 new hires at the Electric Car Company will be split into three groups. Four will be sent to Dallas, Three to Los Angeles and Five to Portland. In how many ways can the group of new hires be divided in this way?

Example A pre-school teacher will split her class of 15 students into three groups with five students in each group. One group will color, a second group will play in the sand box and the third group will nap. In how many ways can the teacher form the groups for coloring, sand box play and napping?

Unordered Partitions

A partition is **unordered** when no distinction is made between subsets of the same size (the order of the subsets does not matter).

We use the "overcounting" principle to find a formula for the number of unordered partitions.

Example Suppose we wish to split our group of 6 friends Alan, Cassie, Maggie, Seth, Roger and Beth into three groups with two people in each group. In this case, we do not have any particular task for each group in mind and we are interested only in finding out how many different ways we can divide the group of 6 into groups of two. In particular the six pairings shown below give us the same unordered partition and is counted only as one such unordered partition or pairing.

AS	CM	RB
CM	AS	RB
AS	RB	CM
CM	RB	AS
RB	AS	CM
RB	CM	AS

The above single unordered partition would have counted as six different ordered partitions if we had a different assignment for each group as in Example A above. Likewise each unordered partition into three sets of two gives rise to 3! ordered partitions and we can calculate the number of unordered partitions by dividing the number of ordered partitions by 3!. Hence a set with 6 elements can be partitioned into 3 unordered subsets of 2 elements in

$$\frac{1}{3!}\binom{6}{2,2,2} = \frac{6!}{3!\ 2!\ 2!\ 2!} = \frac{6!}{3!(2!)^3} \quad \text{ways}$$

In a similar way, we can derive a formula for the number of unordered partitions of a set.

A set of n elements can be partitioned into k **unordered subsets** of r elements each ($kr = n$) in the following number of ways:

$$\boxed{\frac{1}{k!}\binom{n}{r, r, \ldots, r} = \frac{n!}{k!\ r!\ r!\ \ldots\ r!} = \frac{n!}{k!(r!)^k}.}$$

Example In how many ways can a set with 12 elements be divided into four unordered subsets with three elements in each?

Example The draw for the first round of the middleweight division for the Bengal Bouts is about to be made. There are 32 competitors in this division. In how many ways can they be paired up for the matches in the first round?

In any unordered partition where k of the subsets have the same number of elements, we must divide the number of ordered partitions by $k!$ in order to get the number of unordered partitions. It is easier to understand the general method than to work from a formula for the general case.

Example Find the number of partitions of a set of 20 elements into subsets of two, two, two, four, four, three and three. No distinction will be made between subsets except for their size.

Example A math teacher wishes to split a class of thirty students into groups. All groups will work on the same problem. Five groups will have four students, two groups will have three students and two groups will have two students. In how many ways can the teacher assign students to the groups?