Learning Goals

1. Setting up the matrix equation for Massey's ratings for examples with a small number of teams.
2. Solving the matrix equation for Massey's ratings for examples with a small number of teams using R.

Massey's Method.

Chapter 2 from "Who's # 1" [1], chapter available on Sakai.

Massey's method of ranking, was also used in the BCS rankings prior to the changes. It makes use of the point differential in a game. The ratings produced can also be manipulated to produce an offensive and defensive rating for each team, which in turn can be used to produce an offensive and defensive ranking for each team.

For any given game between Team i and Team j, the point differential for team i is the score of team i minus the score for team j. At any given point in a tournament, the **point differential for team i** will be the sum of their point differentials for the games they have played.

Example In our running example, the 2015 six nations rugby cup, we had the following summary of the data up to the end of Round 2:

Feb 25	Ire.	Eng.	Wal.	Scot.	Fra.	It.
Ire.					18-11	26-3
Eng.			21-16			47-17
Wal.		16-21		26-23		
Scot.			23-26		8-15	
Fra.	11-18			15-8		
It.	3-26	17-47				

Notice that the **point differentials** give us a rating of the teams which could be used to obtain a ranking of the teams:

Team	Point Differential
Ireland	30
England	35
Wales	-2
Scotland	-10
France	0
Italy	-53

$\rightarrow$

Team	Rank
Ireland	2
England	1
Wales	4
Scotland	5
France	3
Italy	6

Massey's ratings (which are used to derive his rankings), are based on the simple principle that for each game the difference in the ratings should be equal to the point differential i.e.

$$r_i - r_j = y_k$$

where r_i = rating for team i, r_j = rating for team j and y_k = score for team i - score for team j for game k.

This gives us a linear equation in the unknowns, r_i and r_j for every game played and thus gives us a (very large) system of linear equations.

[1] Who's # 1, *Amy N. Langville & Carl. D. Meyer*, Princeton University Press, 2012.

Example In our example above this would give us 6 equations in the 6 unknowns r_1, r_2, r_3, r_4, r_5 and r_6:

$$\begin{array}{cccccccc}
r_1- & & & & r_5 & & = & 7 \\
r_1- & & & & & r_6 & = & 23 \\
 & r_2- & r_3 & & & & = & 5 \\
 & r_2- & & & & r_6 & = & 30 \\
 & & r_3- & r_4 & & & = & 3 \\
 & & & r_4- & r_5 & & = & \text{-7}
\end{array}$$

In general, if we have n teams, these equations translate to a matrix equation of the form

$$\mathbf{Xr} = \mathbf{y}$$

where the matrix $\mathbf{X}$ is very sparse (lots of 0's) and $\mathbf{r}$ is the column matrix $\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix}$. The matrix $\mathbf{X}$ has a row for each game played and n columns, with only two non-zero entries on each row. The matrix $\mathbf{y}$ is a column matrix, with a row for each game played and entry equal the point differential for that game (sign depends on the order in which the teams appear). Typically this turns out to be an inconsistent system of equations with no solutions for $r_1, r_2, r_3, ..., r_n$. (we showed this for the example above in a previous lecture.)

One can however use the statistical method of least squares to find a solution to the *normal equations* we get by multiplying by the transpose of $\mathbf{X}$, denoted by $\mathbf{X^T}$ (this is the matrix we get by switching the entries $\mathbf{X}_{i,j}$ and $\mathbf{X}_{j,i}$ in the matrix $\mathbf{X}$). The solution to the matrix equation

$$\mathbf{X^T X r} = \mathbf{X^T y}$$

is the best estimate (in a statistical sense of minimizing variance) for the ratings $\mathbf{r}$ in the original equation. The matrix $\mathbf{X^T X}$ is an $n \times n$ matrix because $\mathbf{X^T}$ has n rows and the matrix $\mathbf{X^T y}$ is a column matrix with n rows. Letting $\mathbf{M} = \mathbf{X^T X}$ and $\mathbf{P} = \mathbf{X^T y}$, we get n linear equations in the n unknowns, $r_1, r_2, \ldots, r_n$:

$$\mathbf{Mr} = \mathbf{P}.$$

Example In our example above, the equation $\mathbf{Xr} = \mathbf{y}$ translates to:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \\ 5 \\ 30 \\ 3 \\ -7 \end{pmatrix}$$

We switch the rows and columns of X to get $\mathbf{X^T} =$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

and $\mathbf{M} = \mathbf{X^T X} =$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The matrix $\mathbf{X^T y}$ is

$$\begin{pmatrix} 30 \\ 35 \\ -2 \\ -10 \\ 0 \\ -53 \end{pmatrix}$$

and the equation $\mathbf{Mr} = \mathbf{P}$ looks like

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 30 \\ 35 \\ -2 \\ -10 \\ 0 \\ -53 \end{pmatrix}$$

Notice that the matrix M on the left looks very like the matrix C in the Colley equation (the only difference is that M has the number of games played by each team on the diaginal instead of 2 plus the number of games).

In general if we have n teams the Massey matrix, $\mathbf{M} = \mathbf{X^T X}$ is almost like the Colley matrix from the previous section. It is an $n \times n$ matrix with

$$M_{ii} = t_i = \text{total number of games played by team i}$$

$$M_{ij} = -\#\text{games played by team i and team j.}$$

The matrix $\mathbf{P} = \mathbf{X^T y}$ on the right hand side of the above equation is a column matrix with dimensions $1 \times n$ (where n is the number of teams) and the ith element is the sum of the point differentials for team i for every game played by team i that season.

Unfortunately the matrix $\mathbf{M}$ above is not invertible, so solving the system of equations $\mathbf{Mr} = \mathbf{P}$ using inverses is not an option. To get around this, Massey replaces the n-th row of the matrix $\mathbf{M}$ by a row of 1's and replacing the final row of $\mathbf{P}$ by a zero. This amounts to the requirement that the $r_i's$ add to 0. The new matrix is invertible and and the new system is solvable and **Massey's ratings** are the solutions for $r_1, r_2, \ldots, r_n$ for this system. We demote the adjusted Massey matrix by $\overline{\mathbf{M}}$ and the adjusted point differential column by $\overline{\mathbf{P}}$. The new equation looks like

$$\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$$

and its solution $\mathbf{r}$ gives us the Massey ratings for the teams.

In Summary If there are n teams competing, the Massey matrix is given by $\mathbf{M}$ where

$$M_{ii} = t_i = \text{total number of games played by team i}$$

$$M_{ij} = -\#\text{games played by team i and team j.}$$

The adjusted Massey matrix is given by $\overline{\mathbf{M}}$ where

$$\overline{\mathbf{M}}_{ij} = M_{ii} \quad \text{if} \quad i < n$$

$$\overline{\mathbf{M}}_{nj} = 1 \quad \text{for all j}$$

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix}$$

where r_i represents the unknown Massey rating for team i.

$$\overline{\mathbf{P}} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_{n-1} \\ 0 \end{pmatrix}$$

where P_i denotes the point differential for Team i. **Massey's Ratings** are given by the solution to the system

$$\overline{\mathbf{M}}\mathbf{r} = \overline{\mathbf{P}}$$

Example In our running example, the adjusted equation looks like the following:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 30 \\ 35 \\ -2 \\ -10 \\ 0 \\ 0 \end{pmatrix}$$

The solution is given by

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} 6.91667 \\ 9.58333 \\ 2.41667 \\ -2.75 \\ 2.08333 \\ -18.25 \end{pmatrix}$$

These ratings can be ordered to give us the **Massey Rankings**:

Team	Rank
Ireland	2
England	1
Wales	3
Scotland	5
France	4
Italy	6

Notice that we get a slightly different ranking than that given by the point differential.

Word of Warning: If we have the final results of a round robin tournament where every team has played every other team exactly once, then the Colley rankings will be the same as those given by Wins minus Losses and the Massey rankings will be the same as those given by the point differential. However as we see above, this is not the case for partial results for a tournament collected during the season.

Example In an inter-dorm basketball round robin, the current score data is shown in the following table:

	Badin	Farley	Lyons	McGlinn	Pangborn	Record	Point Differential
Badin		37-82		37-68		0-2	-76
Farley	82-37		64-46	55-47	57-37	4-0	91
Lyons		46-64		37-35	33-60	1-2	-43
McGlinn	68-37	47-55	35-37		44-82	1-3	-17
Pangborn		37-57	60-33	82-44		2-1	45

(a) Set up the matrix equation you must solve to get the Massey ratings $Mr = P$:

(b) Solve the system of equations in R to find a ranking for the teams:

Team	Rank
Badin	
Farley	
Lyons	
McGlinn	
Pangborn	