Mixed Strategies

When the reduced payoff matrix has a single strategy for each player, this gives the optimal strategy for each player, assuming best play by the opponent. We saw however, that some games such as Chicken do not reduce to a single strategy for each player. It is clear in these games that being predictable is a clear disadvantage for wither player, however beyond hiding our intentions, it is unclear how to proceed. In this section, we will study how a player might choose between different strategies if he knows his opponent's strategy. Our main tool will be the expected value.

Lets assume that a player has n possible strategies to choose from. A **mixed strategy** for a player with strategies numbered from 1 to n is a set of numbers (probabilities) $p_1, p_2, \ldots, p_n$ with $0 \leq p_i \leq 1$ for which $p_1 + p_2 + \cdots + p_n = 1$. The player using such a mixed strategy selects the ith strategy with probability p_i. This selection must be random, so that play is not predictable and the selection should be made with the aid of some random device such as a spinner or die. We can represent a mixed strategy for a row player as a row matrix $(p_1, p_2, \ldots, p_n)$. We can represent a mixed strategy for a column player as a column matrix $\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$.

The original strategies are called **Pure Strategies**. A player who plays a single strategy (say strategy i) with probability 1 is said to play a pure strategy or a fixed strategy. This might be taught of as a mixed strategy where $p_i = 1$ and all other p_j's are equal to zero.

Example Suppose that Rory and Connor are playing a game of squash. Let's consider the situation where Rory is about to play the ball and each player has two possible strategies. Rory may play the ball to the front of the court (F) or to the back of the court (B). Connor may anticipate (and move slightly towards) a shot to the front of the court (F) of a shot to the back of the court (B). Rory's pay-off matrix is shown below and gives Rory's chances of winning the point (in percentages) for each possible scenario. We assume that these probabilities are known to both players from experience.

		Connor	
		F	**B**
Rory	**F**	40	70
	B	80	20

If Rory plays the **mixed strategy** $(1/2, 1/2)$. This means he plays to the front of the court half of the time and to the back of the court half of the time in an unpredictable manner. His choice of where to play the ball is random on any given play. This might be done by glancing at a clock an playing to the front if the second hand is in the right half of the clock and playing to the back if the second hand is in the left half of the clock.

If Rory plays the **pure strategy** F, it means he is sure to play to the front.

If Connor play a mixed strategy of $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$, this means that he anticipates a shot to the front of the court 3/4 of the time and a shot to the back of the court 1/4 of the time.

Example Lets consider a previous example where two business', Fitness Indiana and Get Up & Go are both about to set up a gym in one of two neighborhoods. The pay-off matrix below shows the number of customers each business will get in each of the four scenarios. This is not a zero sum game. It is a simultaneous move game and neither player knows which neighborhood will be chosen by the other business. Recall that the reduced matrix here has a single strategy for each player and that is the optimal strategy for each player assuming optimal play by the other player.

		Fitness	**Indiana**
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

Lets assume however that Fitness Indiana is not playing optimally and word has gotten out that there is a 50% chance that they will set up their gym in the First neighborhood (as hence a 50% chance that they will set it up in the second neighborhood). This can be viewed by Get Up & Go as a mixed strategy for Fitness Indiana, namely the strategy

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Using Expected Pay-off to decide between Strategies

When faced with a choice between several strategies, when you know your opponents strategy it is common to choose the strategy yielding the highest expected pay-off. In order to use this method, the pay-off's should have numerical values. Also it does always reflect the most preferred outcome; for example most of us would prefer to have a sure $1 million than a 1/500 chance of getting $1 billion despite the fact that the expected payoff for the second scenario is higher than that for the first.

Recall that if a strategy yields a pay-off of x_i with probability p_i for $1 \leq i \leq n$, the expected payoff for that strategy is

$$\textbf{expected pay-off} = x_1p_1 + x_2p_2 + \cdots + x_np_n.$$

Playing against Known Mixed Strategies

Case 1: Computing the expected Pay-Off to a player who uses a pure strategy against an opponent's mixed strategy.

Here we assume that a player knows his opponent's mixed strategy (or fixed strategy) from observation and wants to choose the best pure strategy. In this case, the player calculates the expected pay-off for each of his pure strategies and chooses the one which yields the highest expected pay-off.

Example Consider the pay-off matrix from the Squash game above between Rory and Connor.

		Connor	
		F	**B**
Rory	**F**	40	70
	B	80	20

Let's assume that from data on previous games, Rory knows that Connor anticipates F with probability 3/4 and B with probability 1/4; in other words, Connor is playing a mixed strategy of $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$. Rory has decided to play a fixed strategy and needs to choose between F and B. He will chose the one which yields the highest expected pay-off.

If Rory plays F, his expected pay-off will be $40(3/4) + 70(1/4) = 47.5$.

(a) Find Rory's expected pay-off if he plays B.

(b) Which pure strategy should Rory play? (assuming that Connor continues to play his mixed strategy of $\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$.)

Note we can calculate these expected payoff's using matrix multiplication. The expected pay-off for each of Rory's fixed strategies above is the product of the corresponding row of the pay-off matrix by the column matrix describing Connor's mixed strategy.

Example Suppose we have the same Pay-off matrix as above and Rory plays a mixed strategy of $(1/2, 1/2)$. What is Connor's optimal pure strategy in this situation?

Here, since we have a constant sum game, and the pay-off's given are for Rory, we can approach the problem in either of two ways. we can convert the pay-off to those for Connor and choose the strategy which gives Connor the highest expected pay-off OR we can use the given matrix and choose the strategy for Connor which gives Rory the lowest expected pay-off.

Example Consider the pay-off matrix for the business' Fitness Indiana and Get Up & Go.

		Fitness	**Indiana**
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

(a) What pure strategy should Get Up & Go Use if Fitness Indiana uses the strategy $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$? (Note this is not optimal play by Fitness Indiana).

(b) What pure strategy should Fitness Indiana use if Get Up & Go use the strategy $(2/3, 1/3)$?

Case 2: Computing the Expected Pay-off to a player when both players use a mixed strategy.

In this case we calculate the expected pay-off for each of the player's pure strategies and then calculate the expected value of the expected values using their mixed strategy. This boils down to a simple matrix calculation

$$\begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix} P \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where P is the pay-off matrix for the given player, $\begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}$ is the mixed strategy for the row player, and $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is the mixed strategy for the column player.

Example Suppose in the above example, Fitness Indiana plays a strategy of $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ and Get Up & Go plays a strategy of $(2/3, 1/3)$.

(a) What is the expected pay-off for Fitness Indiana?

(b) What is the expected pay-off for Get Up & Go?

Example The strategies for a Pitcher and a Batter in Baseball are shown below. We assume the Pitcher uses a strategy of $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ and the batter uses a strategy of $(1/8, 1/2, 1/4, 1/8)$.

		Pitcher		
		Fastball	**Change-up**	**Slider**
Batter	**Fastball**	0.4	0.3	0.35
	Change-up	0.25	0.4	0.4
	Slider	0.2	0.39	0.45
	None	0.3	0.39	0.4

(a) what is the expected pay-off for the Batter?

(b) what is the expected pay-off for the Pitcher?

(c) Would the mixed strategy $(1/4, 1/4, 1/4, 1/4)$ be a better mixed strategy for the Batter, assuming that the pitcher sticks to the strategy $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$?

References

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