Straight Line Motion, Functions, average velocity and speed.

Motion occurs whenever an object changes position. Since objects cannot instantaneously change position, rather they do so progressively over time, time must be a factor in analyzing motion. Below, we will use the mathematical concept of a function to explore motion using various functions of time.

Straight line motion

Functions A function arises when one quantity depends on another. Many everyday relationships between variables can be expressed in this form. In particular, **function** is a map or a rule which assigns to each element x of a set A exactly one element called $f(x)$ in a set B. If the value of a variable, y, depends on the value of another variable, x, y is called the dependent variable and x is called the independent variable.

For example, the distance, d, that a marathon runner has covered (since the beginning of the race) is a function of the amount of time, t, that has passed since the beginning of the race. The set A to which this function can be applied is the set of all times, t, between 0 and the length of time it takes this runner to complete the course. For any one of these times, there is a unique distance, $d(t)$, associated with it. All such distances are between 0 and 26 miles. Therefore we could take $B = [0, 26]$ in the definition above.

Sometimes we can give a single formula for the function, sometimes we need several formulas to describe the function and sometimes we have to rely on partial information about the function in the form of statistics.

Example A: a linear function Lets assume that you walk 5 miles at a constant pace on the track in the Loftus center so that you burn 100 calories per mile. Express calories burned, C, as a function of distance walked, d using a single formula.

Example B: Piecewise defined function Suppose you do a short walk on your treadmill as follows. You walk at
3 miles per hour for 5 minutes
3.5 miles per hour for 10 minutes
4 miles per hour for 10 minutes.
Here the distance travelled, d, is a function of time elapsed since the walk began, t (measured in minutes). However the function is more complicated than the previous example and cannot be expressed with a single formula. Use more than one formula to express d as a function of t. (Note, the speed is given in miles per hour).

Example C The following table shows the distance travelled by a cyclist on a 100 mile cycle at one hour time intervals throughout the cycle. Here, distance travelled, d, is a function of time, t. Using the partial information about that function shown below, estimate the value of d when $t = 2.5$ hours.

$TimePassed(hours)$	0	1	2	3	4	4.5
$Distance(miles)$	0	20	45	65	90	100

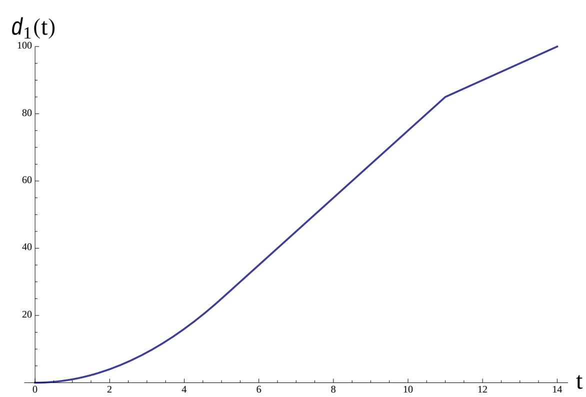
The set of numbers (or objects) to which we apply the function, A, is called the **domain** of the function. The set of values of B which are equal to $f(x)$ for some x in A is called the **range** of f. We have

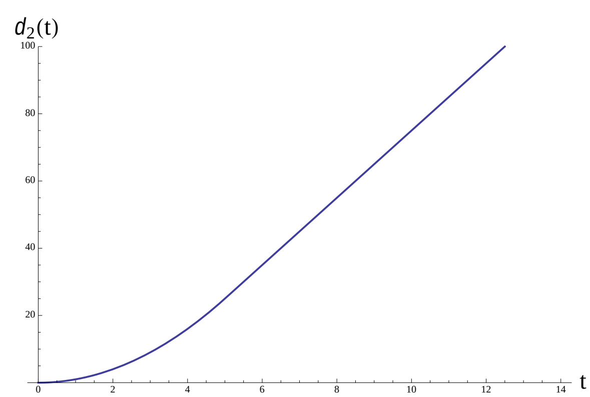
$$\text{range of } f = \{f(x) | x \in A\}$$

Example What is the domain and range of each of the functions in examples A, B and C above?

The **graph** of a function $f(x)$ on the co-ordinate plane is the set of all points (x, y) which satisfy the equation $y = f(x)$. You will find a catalog of graphs of well known functions at the end of this lecture.

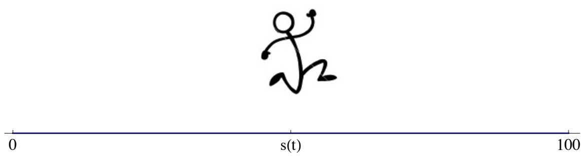
Example Two athletes Matt and Paul are running a 100 meter race. The distances that Matt and Paul have run after t seconds are given below by the functions $d_1(t)$ and $d_2(t)$ respectively. Which runner won the race?

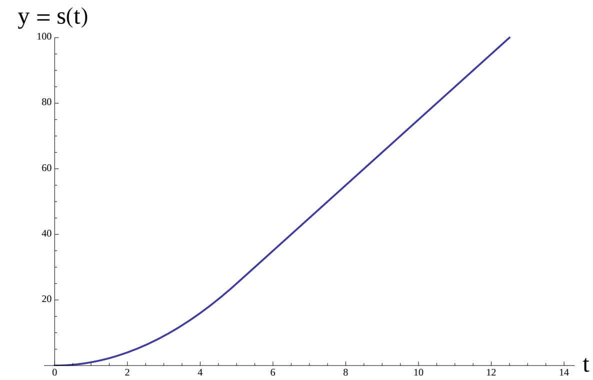




Position The **position** of an object refers to its location in space relative to some reference. If an object is moving in a straight line, we often choose an x-axis as the line along which the motion takes place. The position of the object at time t is given by its x coordinate. The position of the object at time t is usually denoted by $s(t)$ or $x(t)$ and is a function of time. In biomechanics, the unit of measurement is the meter.

Example We might represent the starting position of an athlete running a 100 meter race as 0 on a single axis. With the positive direction of the axis pointing in the direction in which the athlete runs, the position of the athlete at any time t during the race may be represented by a point on the axis, $s(t)$. The picture below shows his position at time t.





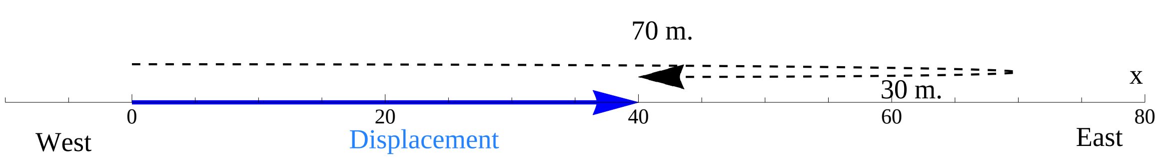
Note that the athlete's position at time t is a function of t and we may draw its graph in a plane with the horizontal axis representing time and the vertical axis representing the dependent variable $y =$ position, as shown on the right.

Distance vs. Displacement We need to make a distinction between the distance an object has travelled and its displacement which is defined as the change in position of the object. That is, displacement tells us how far the object is from its starting point (with direction indicated by the sign in the one dimensional case). To see the distinction between total distance and displacement, imagine a person walking 70 meters to the east and then turning around and walking back a distance of 30 meters to the west. The total distance travelled is 100 m., but the displacement is only 40 m. for the starting point.

Note that in the case of straight line movement, distance travelled is always a positive number whereas displacement may be positive or negative (For example if the person walked 70 m. E and 90 m. west above). It is best to think of displacement as a **vector**, a quantity with both magnitude and direction. Such quantities are represented by arrows in a diagram and we will make full use of them when describing two dimensional motion. In both examples quoted above, we represent the displacement by the blue arrow (vector) and the path of the walker with a dotted line.

Example 1

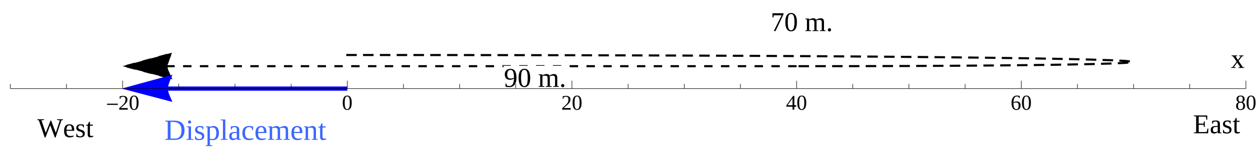
Walks 70 m. E and 30 m. W



Displacement $= +40$

Example 2

Walks 70 m. E and 90 m. W



Displacement = -20

We will discuss vectors in more detail later. For now, since we are dealing with motion in one dimension, the direction of the displacement vector will be indicated by the sign of the displacement. As shown in the examples above, vectors pointing in the positive direction of the given axis of reference will have a positive sign and vectors pointing in the opposite direction will have a negative sign.

In the above examples, displacement measures how far the object is from its starting point. When considering the motion of an object over a particular time interval, say from time t_1 to time t_2, the displacement of the object over that time interval measures how far the object moves from its position at time t_1 over the course of that time interval. If the position of the object at time t_1 is given by x_1 and the position at time t_2 is given by x_2, then the displacement of the object over that time interval is $x_2 - x_1$ and is denoted by

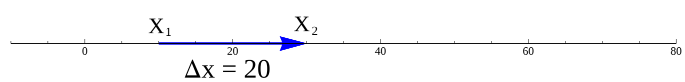
$$\Delta x = x_2 - x_1.$$

In mathematics the Greek letter Δ is often used to denote change. In our previous notation we denoted the position of an object at time t by $s(t)$. With this notation, the displacement over the time interval $[t_1, t_2]$ is denoted by

$$\Delta s = s_f - s_i = s(t_2) - s(t_1),$$

where s_f denotes the final position of the object and s_i denotes the initial position.

Example 4 A runner runs from initial position $x_1 = 10$ m. to a final position at $x_2 = 30$ m. as in the diagram below:

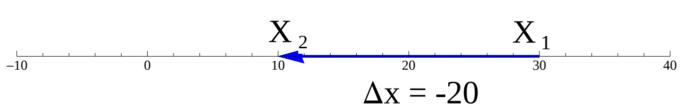


Then

$$\Delta x = x_2 - x_1 = 30 - 10 = 20m.$$

On the other hand

Example 5 If a runner runs from initial position $x_1 = 30$ m. to a final position at $x_2 = 10$ m. as in the diagram below:



Then

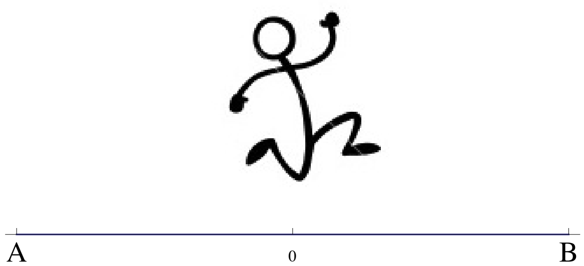
$$\Delta x = x_2 - x_1 = 10 - 30 = -20m.$$

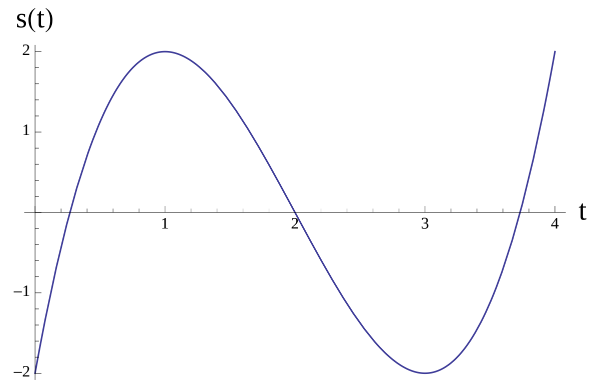
where the negative sign shows that the displacement vector is pointed in the opposite direction to the positive direction of the reference axis.

Since displacement from the initial position is a function of the time that has passed, we can create a displacement function which is often denoted by $\delta(t)$ in books on biomechanics. This function fixes the initial position and measures displacement at time t as the displacement from the starting position (at time $t = 0$).

$$\delta(t) = s(t) - s(0), \quad \text{where } s(t) \text{ is the position of the object at time } t$$

Example 6 An athlete doing agility training starts at point A and runs to point B and then turns and runs back to point A and turns again and runs back to point B. The position function for the athlete at time t is given by $s(t) = t^3 - 6t^2 + 9t - 2$. A graph of the position function is shown below for $0 \le t \le 4$. Let $\delta(t)$ denote the displacement from the initial position as described above and let $d(t)$ denote the distance travelled by the athlete at time t.





What is the starting position of the athlete, $s(0)$?

What is the position of the athlete after 1 second?

What are the values of $\delta(1)$ and $\delta(2)$? Give a verbal explanation of the meaning of these numbers.

What is the distance travelled by the athlete after 2 seconds, $d(2)$?

Sketch the graphs of the functions $\delta(t)$ and $d(t)$ and give a verbal explanation of their meaning.

What is the displacement of the athlete over the time interval from $t = 1$ to $t = 2$?

Average Speed and Velocity If we want to calculate the **average speed** of an object in the time interval $[t_1, t_2]$, we divide "how far" by "how fast". We have

$$\text{Average speed} \;=\; \frac{\text{change in distance}}{\text{change in time}} \;=\; \frac{\Delta d}{\Delta t} = \frac{d(t_2) - d(t_1)}{t_2 - t_1},$$

where $d(t)$ denotes the distance the object has travelled since time $t = 0$. As above, the symbol Δ is used to denote "change in" and should be read as such. Note that average speed is always a positive number.

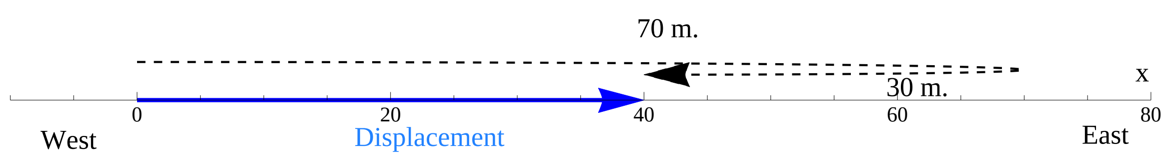
Velocity on the other hand is used to signify both the <u>magnitude</u> of (numerical value) of how fast an object is moving <u>and the direction</u> in which it is moving (determined by the sign + or − for motion in one dimension). Therefore velocity is a vector. We calculate the **average velocity** of an object in the time interval $[t_1, t_2]$ by dividing the change in position (or displacement) by the change in time. We have

$$\text{Average velocity} \;=\; \frac{\text{displacement}}{\text{change in time}} \;=\; \frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}.$$

Note that average velocity may have negative values. A commonly used notation for the average of a variable is to place a bar over the name of the variable. Thus average velocity is commonly denoted by the symbol $\bar{v}$

Example 1 Revisited Suppose the walker in example 1 took 70 seconds to complete their walk of 70 m. E and 30 m. W.

Walks 70 m. E and 30 m. W

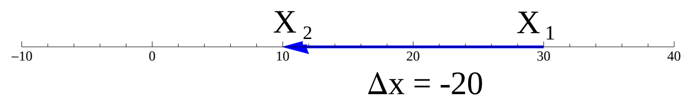


Displacement = +40

(a) What was their average speed for the walk?

(b) What was their average velocity for the walk?

Example 5 Revisited If a runner runs from initial position $s(0) = x_1 = 30$ m. to a final position at $s(5) = x_2 = 10$ m. as in the diagram below, where $s(t)$ denotes the position of the runner at time t seconds after leaving the point x_1, find the average speed and the average velocity of the runner over the time interval $[0, 5]$.



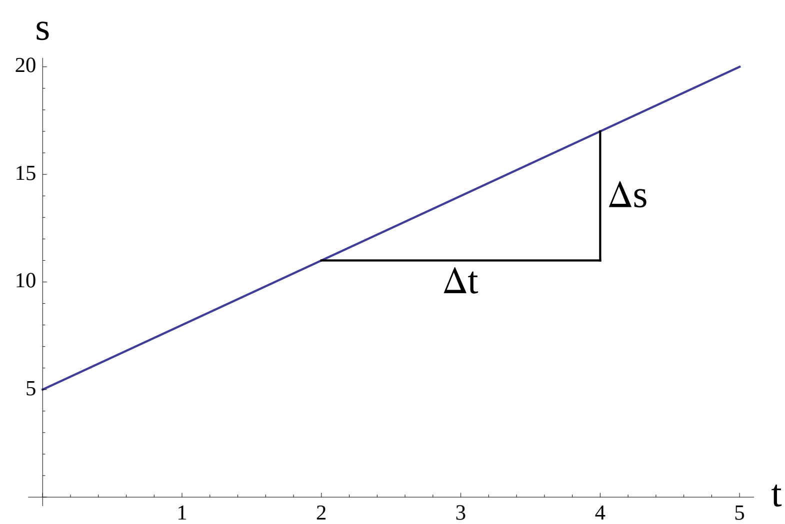
Example 6 revisited Consider the athlete doing agility training above.

(a) What is the average speed of the athlete over the time interval $0 \le t \le 2$ sec. ?

(b) What is the average velocity of the athlete over the time interval $0 \le t \le 2$ sec. ?

Slope of a line Recall that the slope of a line is a constant which can be calculated as the change in vertical position divided by the change in horizontal position. In the picture of the position function below, we have the slope is

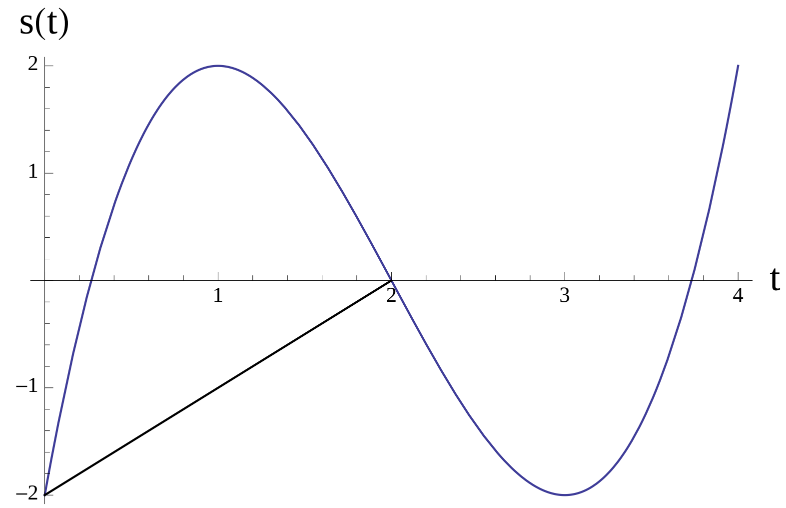
$$m = \text{slope} \;=\; \frac{\Delta s}{\Delta t}.$$



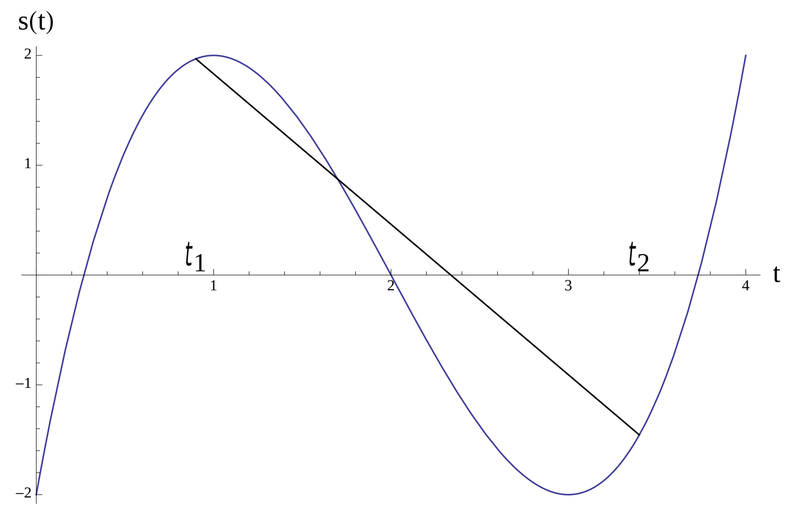
The formula for such a line or linear function is given by $s(t) = mt + b$. Here m is the slope and b is the point where the graph cuts the vertical axis (where $t = 0$).

Average Velocity and secant lines

Note that the average velocity of the athlete in example 6 on the interval $[0,2]$, $\bar{v} = \frac{s(2)-s(0)}{2-0}$, is the slope of the secant line on the graph shown below.



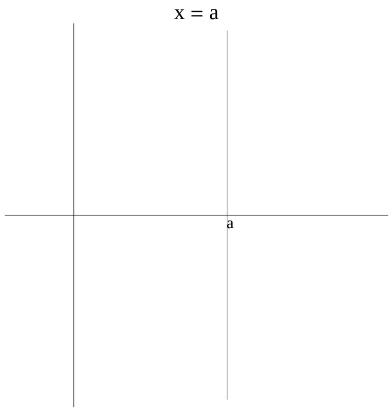
Similarly the average velocity of the athlete over any time interval $[t_1, t_2]$, $\bar{v} = \frac{s(t_2)-s(t_1)}{t_2-t_1}$, is given by the slope of the secant joining the points on the graph of $s(t)$ corresponding to t_1 and t_2.



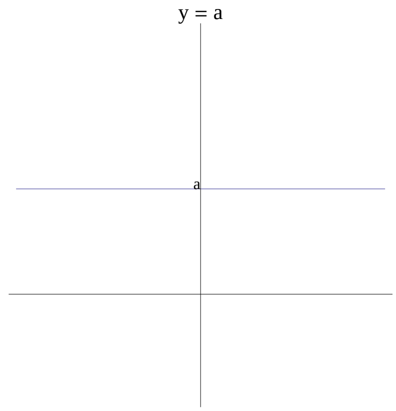
Here is a catalogue of basic functions:

Lines

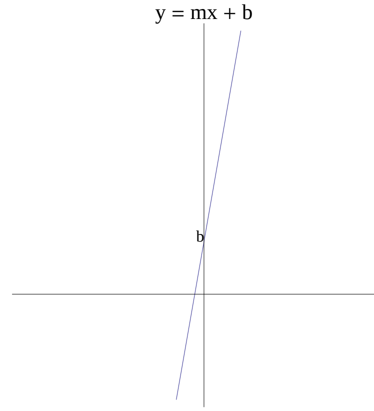
Vertical



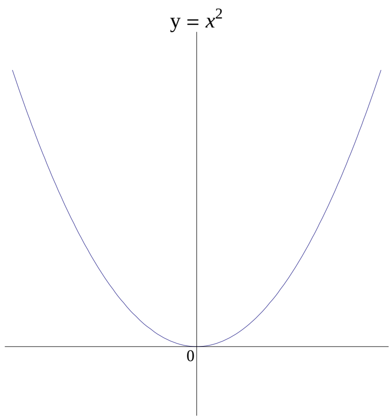
Horizontal

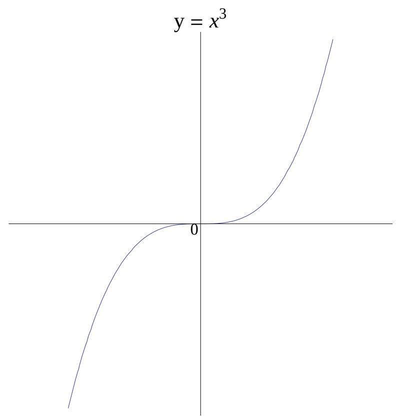


General

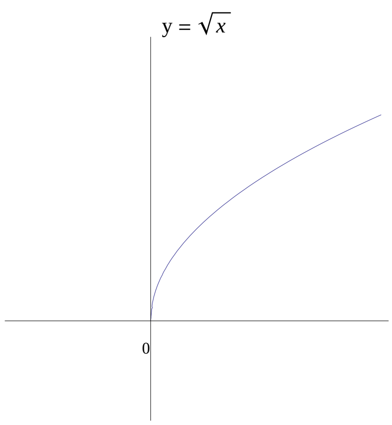


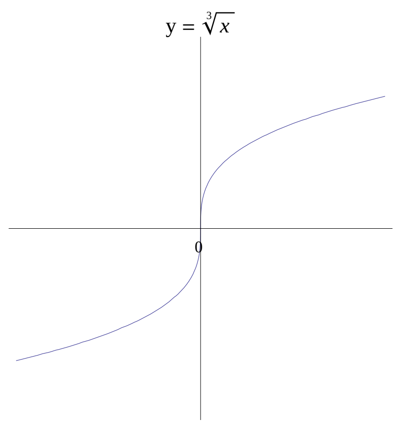
Power Functions



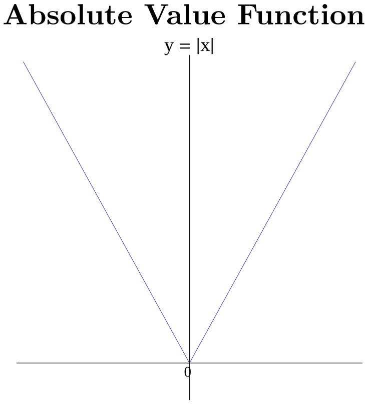


Root Functions





Absolute Value Function



$y = 1/x$