Antiderivatives and motion with constant Acceleration.

Suppose now, we have some information about the initial position and velocity of an object and we know its acceleration throughout the movement we should be able to retrieve information about (or predict) its path or trajectory. In this case, we need to reverse the process of differentiation in order to find the position function. This process is called anti-differentiation or integration.

As it turns out, for any given velocity function, \( v(t) \), there is an infinite family of position functions \( s(t) \) with \( v(t) \) as their derivative. However they are all related to each other by adding a constant, hence once we know a particular one, we can tell what the others look like. More formally:

**An Antiderivative** for a function \( f(t) \) is a function \( F(t) \) with derivative \( F'(t) = f(t) \).

**Example** If \( f(t) = t \), then \( F(t) = \frac{t^2}{2} \) is an antiderivative for \( f(t) \) (Because \( F'(t) = t \). Also \( G(t) = \frac{t^2}{2} + 5 \) is another antiderivative for \( f(t) \). To check this, we simply differentiate \( G(t) \) with respect to \( t \) and check that the derivative also equals \( f(t) = t \).

If \( F(t) \) is a particular antiderivative for a function \( f(t) \), then **every antiderivative for \( f(t) \) is in the family \( F(t) + C \) where \( C \) is a constant.** We use the notation

\[
\int f(t)dt = F(t) + C
\]

to denote this fact.

**Example** We have

\[
\int t \ dt = \frac{t^2}{2} + C
\]

describing the entire family of antiderivatives for \( f(t) = t \). So if the velocity of an object traveling in a straight line at time \( t \) is given by \( v(t) = t \), then its position at time \( t \) is given by \( \frac{t^2}{2} + C \) for some constant.

**Example** What is the position function of an object traveling along a horizontal axis with velocity \( v(t) = t \) and initial position \( s(0) = 2 \).

In a regular calculus course, much attention is given to methods of calculating antiderivatives which we do not have time to pursue. We will need just one or two particular antiderivatives to make predictions about objects moving with constant acceleration. In particular, we will need the formula referred to as **The power rule**:

\[
\int t^n dt = \frac{t^{n+1}}{n}, \quad n \neq -1.
\]

Note that \( t^0 = 1 \), so this applies to the function \( f(t) = 1 \). Just as with differentiation, anti-differentiation is a linear operator, in that

\[
\int a f(t) + b g(t)dt = a \int f(t)dt + b \int g(t)dt.
\]
We also have

\[ \int 0 \, dt = C \]

Note that the velocity function is an antiderivative of the acceleration function of an object and the position function is an antiderivative of the velocity function of an object.

\[
\int a(t) \, dt = v(t) \quad \text{and} \quad \int v(t) \, dt = s(t).
\]

Motion in a straight line with constant acceleration due to Gravity

Suppose a ball is thrown vertically upwards with initial speed \( v(0) = v_0 \) from a height of \( h(0) = h_0 \) meters. In this case, after the ball leaves the throwing hand, (assuming that air resistance has a negligible effect of the path of the ball) the only force acting on the ball is the force of gravity. Hence the acceleration is constant and is approximately equal to \( g = -9.8 \text{m/s}^2 \). Hence the velocity function is given by

\[ v(t) = \int a(t) \, dt = \int (-9.8)t^0 \, dt = -9.8t + C \]

for some constant \( C \). Now letting \( t = 0 \) in this equation gives us the value of the constant: \( v(0) = -9.8(0) + C \rightarrow v_0 = C \). Hence

\[ v(t) = -9.8t + v_0. \]

We can also derive a formula the height (or position) function from the information given:

\[ h(t) = \int v(t) \, dt = \int (-9.8)t + v_0 \, dt = (-9.8)\frac{t^2}{2} + v_0t + C \]

for some constant \( C \). Again we can find this constant by setting \( t = 0 \) in the equation to get \( h(0) = 0 + 0 + C = C \). Thus we get:

\[
h(t) = (-4.9)t^2 + v_0t + h_0.
\]

Example: Ball thrown upward A person throws a ball upwards into the air, with initial velocity of 15.0 m/s from a height of 2 meters.

(a) Give a formula for the height of the ball at time \( t \) seconds after its release

(b) If the ball is not caught on the way down, how long before it reaches the ground?

Note, here you need to use the quadratic formula for the solution to a quadratic equation \( at^2 + bt + c = 0 \) given by

\[ t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
Projectile motion

In this section, we study the motion of objects moving through the air near the earth’s surface, such as a golf ball, basketball, baseball, football or an athlete doing a long or high jump or diving from a platform. Although air resistance is very important, we will not consider it in this section. We will not be concerned with the process by which the object is thrown, rather the path it will follow after it has been projected or launched. Since the only force assumed to be acting on the object after its launch is gravity, which is a constant downward force, we can easily predict the path of the object if we are given its initial speed and the angle at which it is thrown along with its initial height. The following app demonstrates that the path of the object depends only on these three factors.

https://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html

Recall that to track the path of an object in two dimensions, we need to give two co-ordinates for the position of the object at time $t$.

Thus our position function will have two equations, one for the $x$ component of the position of the object at time $t$ and one for the $y$ component. Note that the $x$ component $x(t)$ measures the horizontal distance from the initial point at time $t$ and the $y$ component $y(t)$ measures the height at time $t$. To derive these equations, we split the initial speed into two components, it’s horizontal speed and its vertical speed. For this we need trigonometric functions, which you can find on any calculator.

Recall the definition of $\sin \theta$ and $\cos \theta$ from trigonometry of right angles. If $\theta$ is the angle shown in the right triangle below, we have

$$\sin \theta = \frac{y}{r} = \frac{\text{length of side opposite } \theta}{\text{hypotenuse}}, \quad \cos \theta = \frac{x}{r} = \frac{\text{length of side adjacent to } \theta}{\text{hypotenuse}},$$

Thus If an object is traveling at a speed of $v$ meters per second and is traveling at an angle of $\theta$ to the horizontal axes as shown below, its horizontal movement per second is given by $v_x$ where $v_x/v = \cos \theta$ and its vertical movement per second is given $v_y$ where $v_y/v = \sin \theta$. Thus we have that the object has horizontal and vertical speeds given by

| Horizontal Speed | $v_x = v \cos \theta$ | Vertical Speed | $v_y = v \sin \theta$. |
In our pictures, the angles are greater than 0, in the case where they are negative, \( v_y \) should be interpreted as a velocity where the negative value indicates that the object is falling and its height is decreasing.

To derive our equations for the position of the object at time \( t \), we use the initial speed and the launch angle to find the initial components of velocity.

**Horizontal motion for a projectile launched with initial speed \( v_0 \) at an angle \( \theta \) to the horizontal:**

The initial horizontal velocity is \( v_{x0} = v_0 \cos \theta \) and the initial position is above the origin, hence \( x(0) = 0 \). Since no horizontal forces are assumed to act on the object as it moves we have horizontal acceleration at time \( t \) is \( a_x(t) = 0 \). Thus the velocity function is constant as \( t \) increases and \( v_x(t) = v_{x0} \). Now the horizontal position at time \( t \) is given by

\[
x(t) = \int v_x(t)dt = \int v_{x0}dt = v_{x0}t + C.
\]

By letting \( t = 0 \), we see that \( 0 = x(0) = 0 + C \) and the horizontal position at time \( t \) is given by

\[
x(t) = v_{x0}t = (v_0 \cos \theta)t,
\]

where \( v_0 \) is the initial speed and \( \theta \) is the angle at which the object is launched.

**Vertical motion for a projectile launched with initial speed \( v_0 \) at an angle \( \theta \) to the horizontal:**

Since the only force acting on the vertical position or height of the object at time \( t \) is gravity, using our discussion above we get that the vertical position at time \( t \) is given by

\[
y(t) = h(t) = y(0) + v_{y0}t - 4.9t^2 = y(0) + (v_0 \sin \theta)t - 4.9t^2,
\]

where the initial vertical velocity is \( v_{y0} = v_0 \sin \theta \) and the initial height is given by \( y(0) \).

Thus at time \( t \), the object has position co-ordinates given by:

\[
(x(t), y(t)) = ((v_0 \cos \theta)t, y(0) + (v_0 \sin \theta)t - 4.9t^2)
\]

assuming that \( x(0) = 0 \), the initial speed is given by \( v_0 \) and the angle at which the object is launched is \( \theta \). Just as the position function for the object has a vertical and a horizontal component at time \( t \), so does its velocity function. We have

\[
\text{horizontal velocity at time } t = x'(t) = v_x(t) = v_0 \cos \theta,
\]
vertical velocity at time $t$  
$$y'(t) = v_y(t) = v_0 \sin \theta - 9.8t.$$  
We get the maximum height occurs where $y'(t) = v_y(t) = 0$ and the object hits the ground when $y(t) = 0$.

**Example** A football is kicked (from the ground) at an angle of $37^\circ$ with a velocity of $20$ m/s. Calculate 
(a) The components of the initial velocity, $v_{x0}$ and $v_{y0}$.

(b) The maximum height the ball reaches (note this happens when the vertical velocity equals 0).

(c) How far away will the ball hit the ground?

**The Range of the projectile (distance before it hits ground).** The time it takes before the projectile hits the ground, $t_f$, is given by the time we get when we set $y(t) = 0$, that is when 

$$y(t) = -4.90t^2 + v_{y0}t + y_0 = 0,$$

or 

$$t_f = \frac{-v_{y0} \pm \sqrt{(v_{y0})^2 + 4(4.9)y_0}}{-2(4.90)}$$
We have the

\[
\text{Range} = x(t_f) = v_{x0}t_f = v_{x0} \left[ \frac{-v_{y0} \pm \sqrt{(v_{y0})^2 + 4(4.9)y_0}}{-2(4.9)} \right]
\]

Note that the range depends on both the take off speed and the angle of take-off.

Not that if \( y_0 = 0 \), this reduces to

\[
t_f = \frac{-v_{y0} \pm \sqrt{(v_{y0})^2}}{-2(4.9)} = \frac{-v_{y0} \pm |v_{y0}|}{-9.8} = 0 \text{ or } \frac{2v_{y0}}{9.8}
\]

So

\[
\text{if } y_0 = 0, \text{ then } t_f = \frac{2v_{y0}}{9.8}.
\]

and

the range is given by \( x(t_f) = v_{x0}t_f = \frac{2v_{x0}v_{y0}}{9.8} = \frac{2(v_{x0}^2 + v_{y0}^2)\cos \theta \sin \theta}{9.8} = \frac{(v_{x0}^2 + v_{y0}^2)\sin(2\theta)}{9.8} \).

where \( \theta \) is the angle at which the projectile is launched.

---

**Angle giving Maximum Range**

If \( y_0 = 0 \), that is the initial height is at ground level, for any given speed, the maximum range is achieved when \( \theta = 45^\circ \). The lower the take off speed, the lower the range and vice-versa.

**The Long Jump** If all combinations of speed and angles were possible for someone competing in the long jump, it seems that a jumper should take off at \( 45^\circ \). However it turns out that a long jumper can generate much higher take off speeds at lower angles and world class long jumpers usually take off at an angle of about \( 21^\circ \).

**Example** Compare the range of a long jump where

(a) the jumper takes off at an angle of \( 21^\circ \) and a speed of \( 9.45 m/s \)

(b) long jump where the jumper takes off at an angle of \( 45^\circ \) and a speed of \( 7.5 m/s \).

In addition to the decrease in the speed of take-off achievable by the jumper at higher angles, the assumption of setting the take-off height to zero is actually incorrect, since the equations for the path of a projectile derived above apply to the center of mass of the object in motion. The initial position of the center of mass of a world class jumper is about 1.2 to 1.3 meters above the ground. This gives
extra range to the jump and also by starting with the center of mass in front of the foot from which they are pushing off, the jumper can increase the range of the jump.

For more details see the following article:
http://people.brunel.ac.uk/~spstnpl/Publications/Ch24LongJump(Linthorne).pdf
and watch the video on “Maximizing The Long Jump of Brian Clay” on NBC LEARN’s series on the Science of the Summer Olympics.

http://www.nbclearn.com/portal/site/learn/science-of-the-summer-olympics

If the initial height is greater than zero then the angle maximizing range is less than 45°. If, for example, the initial height is say 2m (as with the shot putt) the angle is smaller, (in this case about 37°.)

Example (a) Use the following Wolfram Alpha application to find the angle giving the maximum range for the shot putt if it is thrown from a height of two meters at 10m/s.

http://demonstrations.wolfram.com/ProjectileMotion/

Once again however the feasible speeds and angles for humans are constrained and typical release angles for world class shot-putters are around 37°. Check for more details on this page:

http://people.brunel.ac.uk/~spstnpl/BiomechanicsAthletics/ShotPut.htm

Center of Mass As mentioned above the equations derived apply to the center of mass of the moving object. If an object has a line or plane of symmetry, the center of mass will be on that line or in that plane (assuming that the object is roughly equally dense throughout and mass is distributed evenly). This sometimes gives us a rough idea of where the center of mass is for an object. For humans standing upright the center of mass is usually in the lower abdomen. However, the position of the center of mass can change as the shape of the object changes and can sometimes lie outside the body. One classic example of where this is manipulated in sports is the Fosbury flop for the high jump. In this jumping technique the athlete arches their shoulders back and legs and keeps a large part of their body below the bar at all times as they jump. The center of mass can remain as much as 20 cm below the bar throughout the jump. Thus it enables the achieve a higher jump height than the maximum height achieved by the center of mass in the jump. Mr. Fosbury won a gold medal in the 1968 Summer Olympics using this technique. It soon became the dominant style and currently remains so.
The fact that the equations apply to the center of mass of a moving body and that the position of the center of mass can change with the shape of the body is also the foundation of another illusion common in sports. The point in time when the maximum height of the trajectory of an object is reached is a single moment in time. However some athletes seem to defy gravity and stay at the maximum height for a prolonged period of time as with basketball players when taking a shot or that a ballerina perfuming a grand jete. When an athlete raises their legs as with the basketball player and the ballerina in the picture, the center of mass continues to rise while the upper body remains at the same height creating the illusion of flying.

More Wolfram Apps to Play with for the armchair athlete.

Baseball

http://demonstrations.wolfram.com/ThrowingABaseballFromTheOutfieldToHomePlate/

Golf

http://demonstrations.wolfram.com/FlightOfAGolfBall/

Discus


Basketball

http://demonstrations.wolfram.com/HoopDreams/
Pole Vaulting

http://demonstrations.wolfram.com/OlympicPoleVaulting/

Soccer

http://demonstrations.wolfram.com/TrajectoryOfASoccerBall/

and more great video links can be found here:

http://www3.nd.edu/~apilking/Math10170S14/Links/Simulations/NewsLInks.htm