Using matrix inverses and Mathematica to solve systems of equations
(Using 2.4, Goldstein, Schneider and Siegel and Mathematica(available on the OIT website))

Given a system of linear equations in two unknowns

$$\begin{cases} -2x + 4y &= 2 \\ -3x + 7y &= 7 \end{cases}$$

we can write it in matrix form as a single equation $AX = B$, where

$$A = \begin{pmatrix} -2 & 4 \\ -3 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

When we multiply we get

$$AX = \begin{pmatrix} -2x + 4y \\ -3x + 7y \end{pmatrix}$$

a 2×1 matrix. When we identify this matrix with the matrix $B = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$., we get two equations equating the elements of each matrix, thus getting our linear system back again: Given a system of linear equations in two unknowns

$$\begin{cases} -2x + 4y &= 2 \\ -3x + 7y &= 7 \end{cases}$$

We can solve this system of equations using the matrix identity

$$AX = B,$$

if the matrix A has an inverse. Namely, we can use matrix algebra to multiply both sides of the equation by A^{-1}, thus getting

$$A^{-1}AX = A^{-1}B.$$

Since $A^{-1}A = I_{2\times 2}$, we get

$$I_{2\times 2}X = A^{-1}B, \quad \text{or} \quad X = A^{-1}B.$$

Lets see how this method works in our example.

Example In our example, we converted the system of equations

$$\begin{cases} -2x + 4y &= 2 \\ -3x + 7y &= 7 \end{cases}$$

to matrix form

$$\begin{pmatrix} -2 & 4 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

Recall to find the inverse of the matrix $A = \begin{pmatrix} -2 & 4 \\ -3 & 7 \end{pmatrix}$, we first find its determinant, which is

$$d = (-2)7 - (-3)4 = -14 + 12 = -2.$$

Since the determinant is non zero, the matrix is invertible and

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 7 & -4 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -7/2 & 2 \\ -3/2 & 1 \end{pmatrix}$$

Mutiplying the above equation by A^{-1}, we get

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7/2 & 2 \\ -3/2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix}.$$

Performing the cancellation on the left and the multiplication on the right, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7/2 & 2 \\ -3/2 & 1 \end{pmatrix}\begin{pmatrix} (-7/2)2 + (2)7 \\ (-3/2)2 + (1)7 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

and our solution to the system is

$$x = 7, \quad y = 4.$$

Example Solve the following system of equations using the matrix approach shown above.

$$\begin{cases} x + 2y & = & 4 \\ -x + 3y & = & 3 \end{cases}$$

The same approach can be used for systems of equations with any number of variables as long as the inverse of the matrix A exists. This happens only when there is a unique solution and the number of variables is equal to the number of equations in the system. In this case, the system of n equations

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n & = & b_2 \\ & \vdots & & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n & = & b_m \end{cases}$$

can be written in the form $AX = B$ where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We then calculate A^{-1} and multiply the equation $AX = B$ by the matrix A^{-1} to get and equation of the form $X = A^{-1}B$ and specific values for $x_1, x_2, \ldots, x_n$.

Example Convert the system of linear equations shown below to a matrix equation of the form $AX = B$.

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + x_4 & = & 2 \\ x_1 + x_2 + 0x_3 + 2x_4 & = & 4 \\ 0x_1 + x_2 + x_3 + 0x_4 & = & 5 \\ 3x_1 + 0x_2 + x_3 + 2x_4 & = & 2 \end{cases}$$

If the determinant of an $n \times n$ matrix, A, is non-zero, then the matrix A has an inverse matrix, A^{-1}. We will not study how to construct the inverses of such matrices for $n \geq 3$ in this course, because of time constraints. One can find the inverse either by an algebraic formula as with 2×2 matrices or using a variation of Gauss-Jordan elimination. In this course, we will use the software package *Mathematica* to find inverses of matrices if we need to and to solve matrix equations such as those shown above.

Mathematica

You may and should download your own free copy of *Mathematica* from the OIT website Go to the OIT homepage
http://oit.nd.edu/
    click on `Software Downloads` in the menu on the left.
Enter your Notre Dame netID and Password in the fields provided
Look for `Mathematica` on the list of available software
Click on `More Details` and follow the directions to download the Student Version.

When you open *Mathematica*, you should open a new Notebook by clicking on `File` in the menu bar, then follow the path `File/New/Notebook`. You can also create a new Notebook, by pressing **Command-n** on your keyboard.
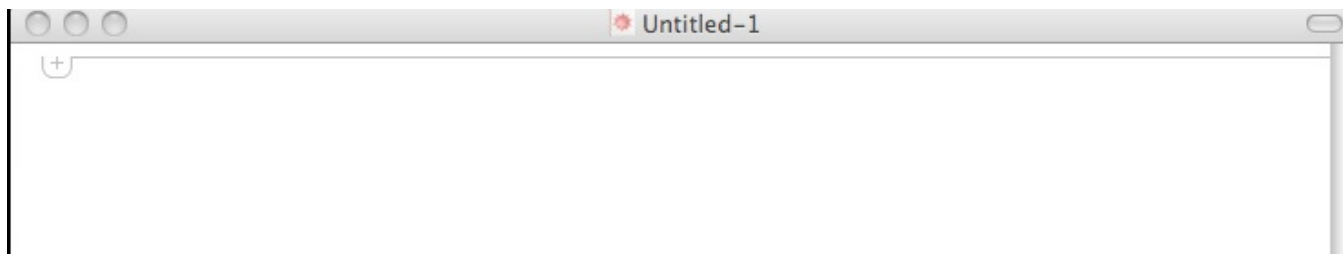
To enter a command, click on the plus mark in the upper left of your document and choose `Mathematica input` from the drop down menu. Note that this is also the default mode, so you can just start typing your command without this step, if you click below the plus mark.

type `2 + 3`
    and press `Shift-Return` to get your answer 5 on a new line.
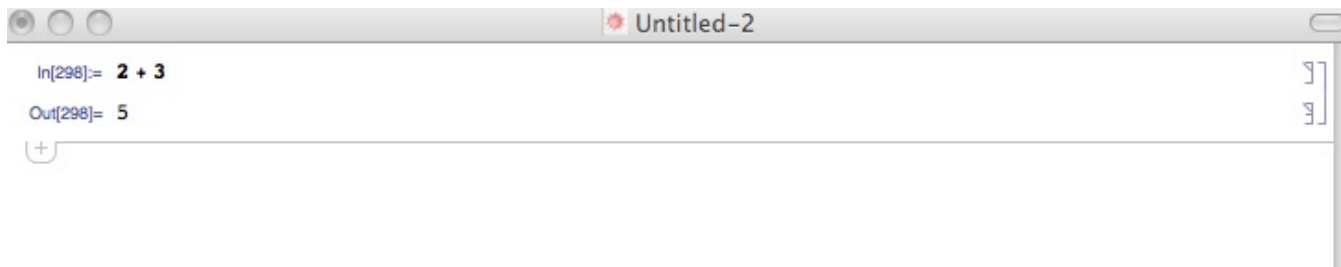
Another way to Evaluate a command in an input cell is to click anywhere on the code in the cell, go to `Evaluation` in the menu at the top of your screen and choose `Evaluate Cells` from the drop down menu.

To solve a system of linear equations in `Mathematica`, we will use their matrix form.

We first learn how to **enter a matrix** and give it a name. We represent the matrix as a list of rows separated by commas between curly brackets. Each row is enclosed by curly brackets with commas separating the entries.

Example The matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 5 & 10 \end{bmatrix}$$

should be entered as

$$\{\{2, 3, 4\}, \{1, 1, 1\}, \{1, 5, 10\}\}$$

When you enter this matrix as input, press `Shift-return` to get

The output, looks exactly like the input. if you prefer the usual matrix format, we will discuss how to view that below. If we want to use this matrix in calculations, we must be able to refer to it, so we must give it a name. To call the matrix m, we can go back to the input cell and simply insert $m =$ in front of our matrix. Then we press `Shift-Return` to get the same output, except now Mathematica knows that this matrix is called m. If you wish to view your matrix in the usual format, you can type the command `MatrixForm[m]` and Mathematica will present the matrix in its usual form.

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In[299]:= **2 + 3**

Out[299]= 5

In[301]:= **m = {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}**

Out[301]= {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}

In[302]:= **MatrixForm[m]**

Out[302]//MatrixForm=

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 5 & 10 \end{pmatrix}$$

Now suppose (as we already have done in an example above), we wanted to solve the system with augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 9 \\ 1 & 1 & 1 & 3 \\ 1 & 5 & 10 & 16 \end{array}\right]$$

then we need to solve for the matrix X in the equation

$$mX = \begin{bmatrix} 9 \\ 3 \\ 16 \end{bmatrix}$$

Lets call the matrix on the right b and enter it into Mathematica. Note there is just one entry in each row.

In[301]:= **m = {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}**

Out[301]= {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}

In[302]:= **MatrixForm[m]**

Out[302]//MatrixForm=

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 5 & 10 \end{pmatrix}$$

In[303]:= **b = {{9}, {3}, {16}}**

Out[303]= {{9}, {3}, {16}}

In[304]:= **MatrixForm[b]**

Out[304]//MatrixForm=

$$\begin{pmatrix} 9 \\ 3 \\ 16 \end{pmatrix}$$

Now to solve the system, we must solve for the matrix X the matrix equation $mX = b$. To do this in Mathematica, we use the command `LinearSolve[m,b]`.

```
In[301]:= m = {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}

Out[301]= {{2, 3, 4}, {1, 1, 1}, {1, 5, 10}}

In[302]:= MatrixForm[m]

Out[302]//MatrixForm=
        ( 2  3   4  )
        ( 1  1   1  )
        ( 1  5  10  )

In[303]:= b = {{9}, {3}, {16}}

Out[303]= {{9}, {3}, {16}}

In[304]:= MatrixForm[b]

Out[304]//MatrixForm=
        (  9 )
        (  3 )
        ( 16 )

In[305]:= LinearSolve[m, b]

Out[305]= {{1}, {1}, {1}}
```

The output is the matrix X, we can view this matrix in matrix format if we change our command to `MatrixForm[LinearSolve[m,b]]`. We se that

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To save your file, use `command-S`. The first time you save the file, you will need to supply a name and choose where to save it.

Example   Convert the system of linear equations shown below to a matrix equation of the form $AX = B$ and solve the system using Mathematica.

$$\begin{cases} x + 2y + z & = & 2 \\ x - y + 2z & = & 4 \\ x - y - z & = & 5 \end{cases}$$

Example Convert the system of linear equations shown below to a matrix equation of the form $AX = B$ and solve the system using Mathematica.

$$\begin{cases} 2x_1 + 4x_2 + 3x_3 + x_4 & = & 2 \\ x_1 + x_2 + 0x_3 + 2x_4 & = & 4 \\ 0x_1 + x_2 + x_3 + 0x_4 & = & 5 \\ 3x_1 + 0x_2 + x_3 + 2x_4 & = & 2 \end{cases}$$