Colley's Method.

Chapter 3 from "Who's # 1" [1], chapter available on Sakai.

Colley's method of ranking, which was used in the BCS rankings prior to the change in the system, is a modification of the simplest method of ranking, the winning percentage. Colley's method gives us a rating for each team, which we can use to find a ranking for the teams.

Note A **ranking** refers to a rank-ordered list of teams and a **rating** gives us a list of numerical scores. Every rating gives us a ranking for the teams.

Winning Percentage Let t_i be the number of times Team i has played and let w_i be the number of times Team i has won, then using the winning percentage, the rating for Team i is given by

$$r_i = \frac{w_i}{t_i}.$$

Dealing with ties: You may ignore ties completely when calculating t_i, w_i and l_i or you could count a tie between the two teams as one game and a half of a win and a half of a loss for each. The important thing in in the derivation of the Colley ratings is that $t_i = w_i + l_i$ for each team. To illustrate the theory behind Colley's rankings in our class example below, we will use the latter method since we have so many ties.

Some of the disadvantages of using this method to rate teams are:

- Ties in the ratings often occur.
- The strength of the opponent is not factored into the analysis.
- At the beginning of the season, the numbers do not make sense, since the ranking for each team is $\frac{0}{0}$.
- As the season progresses, a team with no wins has a rating of $\frac{0}{t_i} = 0$.

Laplace's Rule The main idea behind Colley's method starts with replacing the winning percentage by a slight modification of it called Laplace's rule of succession. The rating for Team i is then given by

$$r_i = \frac{1 + w_i}{2 + t_i}.$$

Colley uses an approximation to this rating to get a system of Linear equations from which he derives ratings r_i for the teams which incorporate strength of schedule.

Example Lets calculate the ratings and rankings given by the winning percentage and Laplace's at various stages of our class pong tournament.

(a) Here are the results for the first two rounds, fill in the winning percentage and the ratings from Laplace's rule in the table below. Find the corresponding rankings.

Pong Tournament (Round Robin): Results for Rounds 1 and 2

[1] Who's # 1, *Amy N. Langville & Carl. D. Meyer*, Princeton University Press, 2012.

Round 1

Player 1	Emily Aberle	1	vs.	Player 6	Danielle Stefania	1
Player 5	Jubril Dawodu	1	vs.	Player 2	Mark Miclean	1
Player 3	Colin Rahill	2	vs.	Player 4	Josh Dunlap	3

Round 2

Player 5	Jubril Dawodu	1	vs.	Player 3	Colin Rahill	0
Player 2	Mark Miclean	1	vs.	Player 1	Emily Aberle	0
Player 6	Danielle Stefania	2	vs.	Player 4	Josh Dunlap	1

After Round 2

i	Team i	$r_i = \frac{w_1}{t_i}$	Ranking Win. %	$r_i = \frac{1+w_i}{2+t_i}$.	Ranking Laplace
1	Emily Aberle				
2	Mark Miclean				
3	Colin Rahill				
4	Josh Dunlap				
5	Jubril Dawodu				
6	Danielle Stefania				

(b) Use the final results shown below to fill in the winning percentage and the ratings from Laplace's rule in the table below. Find the corresponding rankings.

Player	**Name**	**W = Wins**	**L = Losses**	**W-L**	**W + L**
1	Emily Aberle.	2.5	2.5	0	5
2	Mark Miclean	4.5	0.5	4	5
3	Colin Rahill	0.5	4.5	-4	5
4	Josh Dunlap	1	4	-3	5
5	Jubril Dawodu	4	1	3	5
6	Danielle Stefania	2.5	2.5	0	5

After Final Round

i	Team i	$r_i = \frac{w_1}{t_i}$	Ranking Win. %	$r_i = \frac{1+w_i}{2+t_i}$.	Ranking Laplace
1	Emily Aberle				
2	Mark Miclean				
3	Colin Rahill				
4	Josh Dunlap				
5	Jubril Dawodu				
6	Danielle Stefania				

Note When using Laplace's rule, all teams/competitors start out with a rating of $1/2 = \frac{1+0}{2+0}$. The ratings move above and below 1/2 as the season progresses. Because one team's/competitor's win is another's loss, the r_i's are interdependent.

Notation, O_i: In the calculations below, we are going to denote the list of teams/competitors that team i has played so far by O_i. This list will vary depending on where we are at in the tournament. If team/competitor i has played team/competitor j twice, then team/competitor j should appear twice on the list.
After round two in our class tournament, the results of which are shown above, we see that competitor 4 is Josh Dunlap., so at that point in the game

$$O_4 = \{\text{Player 3; Colin Rahill, Player 6; Danielle Stefania}\}.$$

After the final round

$$O_4 = \{\text{Player 3; Colin Rahill, Player 6; Danielle Stefania, Player 1; Emily Aberle,}$$

$$\text{Player 2; Mark Miclean, Player 5; Jubril Dawodu}\}.$$

We will continue to use the notation t_i for the number of games team/competitor i has played so far in the tournament. At any point in the tournament. t_i will equal the number of elements on the list O_i.

After round two in our class tournament, $t_4 = 2$ and after the final round , $t_4 = 5$.

An approximation for $\frac{t_i}{2}$: Colley makes an approximation shown in our calculations below which uses the fact that the Laplace ratings for each team fluctuate around 1/2 and that they are interdependent, so one might expect them to average out to 1/2 for a large number of teams in the tournament. The approximation says that if I sum the Laplace ratings over all of the teams played by team i at some given points in the tournament, the that sum should be roughly equal to one half of the number of games played by team i at that point in the tournament, specifically:

$$\frac{t_i}{2} = \sum_{k=1}^{t_i} \frac{1}{2} \approx \sum_{k \in O_i} r_k$$

where r_k denotes the current ranking for team/competitor K.

In our example above After round two

$$\sum_{k \in O_4} r_k = r_3 + r_6 = 1/4 + 5/8 \text{ (since the Laplace rankings for both are 1/2 at this point).}$$

of course $\frac{t_i}{2} = \frac{2}{2} = 1$ so the approximation is reasonable.

In our example above After the final round

$$\sum_{k \in O_4} r_k = r_1 + r_2 + r_3 + r_5 + r_6 = \frac{7}{14} + \frac{11}{14} + \frac{3}{14} + \frac{10}{14} + \frac{7}{14} = \frac{38}{14} \approx 2.71.$$

On the other hand $\frac{t_i}{2} = \frac{5}{2} = 2.5$, so the approximation is not exact in this case but is **a reasonable approximation.**

Colley Ratings Colley's ratings use the above approximation to the Laplace ratings to derive a system of linear equations. The solution to this system give Colley's ratings, which in turn give Colley's rankings. Let t_i denote the number of games Team i has played, let w_i denote the number of games that Team i has won (we consider a draw as half a win and half a loss) and l_i, the number of games they have lost. Let r_i denote the ratings we get using Laplace's rule. We have

$$\begin{aligned} w_i &= \frac{w_i - l_i}{2} + \frac{w_i + l_i}{2} \\ &= \frac{w_i - l_i}{2} + \frac{t_i}{2} \\ &= \frac{w_i - l_i}{2} + \sum_{k=1}^{t_i} \frac{1}{2} \end{aligned} \tag{0.1}$$

Since all teams begin with $r_k = \frac{1}{2}$ and the ratings are distributed around this number as the season progresses, we have an approximation

$$\sum_{k=1}^{t_i} \frac{1}{2} \approx \sum_{k \in O_i} r_k$$

where O_i denotes the set of teams that have played team i. Thus the ratings we get from Laplace's rule, $\{r_i\}$, approximately satisfy the system of n equations (n = the number of teams):

$$w_i = \frac{w_i - l_i}{2} + \sum_{k \in O_i} r_k. \tag{0.2}$$

By definition, the ratings from Laplace's rule satisfy

$$r_i = \frac{1 + w_i}{2 + t_i}.$$

Substituting for w_i from Equation 0.2, we get a system of equations which are approximately satisfied by the ratings we get from Laplace's rule:

$$r_i = \frac{1 + \frac{w_i - l_i}{2} + \sum_{k \in O_i} r_k}{2 + t_i}.$$

Multiplying Equation i across by $2 + t_i$ we get

$$(2 + t_i) r_i = 1 + \frac{w_i - l_i}{2} + \sum_{k \in O_i} r_k$$

and subtracting $\sum_{k \in O_i} r_k$ from both sides of Equation i, we get

$$(2+t_i)r_i - \sum_{k \in O_i} r_k = 1 + \frac{w_i - l_i}{2}. \tag{0.3}$$

It can be shown that this system of equations has a unique solution and **Colley's ratings** are the ratings which actually satisfy these equations. if we have n teams in a conference, we can write this system of equations in Matrix form as

$$\begin{pmatrix} 2+t_i & -n_{12} & -n_{13} & .... & -n_{1n} \\ -n_{21} & 2+t_2 & -n_{23} & .... & -n_{2n} \\ \vdots & \vdots & \vdots & .... & \vdots \\ -n_{n1} & -n_{n2} & -n_{n3} & .... & 2+t_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where $b_i = 1 + \frac{w_i - l_i}{2}$ and n_{ij} denoted the number of times Team i and Team j have faced each other.

In summary, we have :

Colley's Method

Colley's ratings are the solutions $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ of the linear system

$$C\mathbf{r} = \mathbf{b}$$

where
C is an $n \times n$ matrix called the Colley Matrix where

$$C_{ij} = \begin{cases} 2+t_i & i = j \\ -n_{ij} & i \neq j \end{cases}$$

t_i	total number of games played by team i
n_{ij}	number of times team i faced team j
$\mathbf{b}_{n\times 1}$	$n \times 1$ matrix on the right with $b_i = 1 + \frac{1}{2}(w_i - l_i)$
w_i	total number of wins accumulated by team i.
l_i	total number of losses accumulated by team i.
$\mathbf{r}_{n\times 1}$	general rating vector produced by the Colley system.
n	number of teams in the conference = order of C.

Example Write out the matrix equation $C\mathbf{r} = \mathbf{b}$ for the Colley method for our class tournament after round 2 and solve the system of equations using Mathematica. Derive the corresponding Colley Rankings for the competitors after round 2.

Example Let's look at some data from 2013 NCAA Mens Basketball, Division 1. Below we look at the data from the games played in the America East conference from Jan 02, 2013 to Jan 10 2013. This data can be found on the ESPN website. The teams in the conference are as follows:

i	Team i	Abbreviation
1	Stony Brook	STON
2	Vermont	UVM
3	Boston University	BU
4	Hartford	HART
5	Albany	ALBY
6	Maine	ME
7	Univ. Maryland, Bal. County	UMBC
8	New Hampshire	UNH
9	Binghampton	BING

The following is a record of their games and results (W/L) from Jan 02, 2013 to Jan 10, 2013:

Date	Teams	Winner
Jan 02, 2013	BING vs HART	HART
Jan 02, 2013	UVM vs UNH	UVM
Jan 02, 2013	BU vs ME	ME
Jan 02, 2013	ALBY vs UMBC	ALBY
Jan 05, 2013	STON vs UNH	STON
Jan 05, 2013	UVM vs ALBY	UVM
Jan 05, 2013	BU vs HART	HART
Jan 05, 2013	ME vs UMBC	ME
Jan 07, 2013	BING vs ALBY	ALBY
Jan 08, 2013	UVM vs BU	BU
Jan 09, 2013	BING vs STON	STON
Jan 09, 2013	ME vs HART	HART
Jan 09, 2013	UMBC vs UNH	UMBC

Example Write out the matrix equation $C\mathbf{r} = \mathbf{b}$ for the Colley method for this example on Jan 09. Solve for the ratings using Mathematica and convert to the Colley rankings.

i	Team i	Abbreviation	Colley Rank
1	Stony Brook	STON	
2	Vermont	UVM	
3	Boston University	BU	
4	Hartford	HART	
5	Albany	ALBY	
6	Maine	ME	
7	Univ. Maryland, Bal. County	UMBC	
8	New Hampshire	UNH	
9	Binghampton	BING	

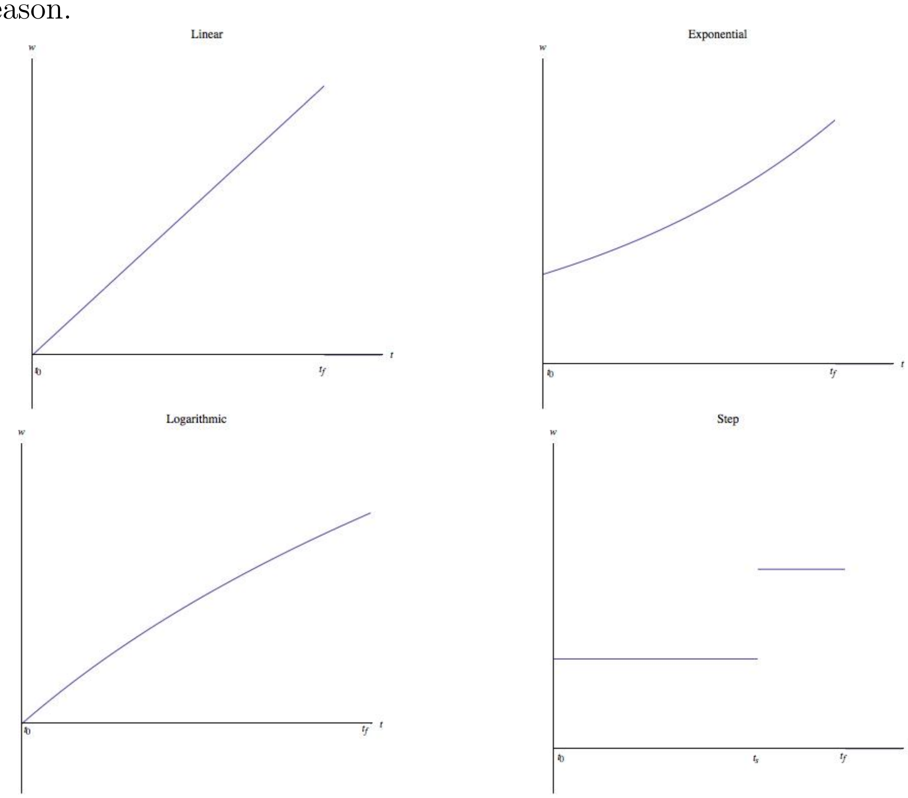
Properties of Colley Ratings

- The Colley ratings are generated using win loss information only. Hence they are unaffected by teams that purposefully run up the score against weak opponents. If one wishes to take the point differential into account, one can use the Massey method (see Who's Number One).
- Each team starts with a rating of $\frac{1}{2}$ and as the season progresses, the ratings bounce back and forth above and below $\frac{1}{2}$. The average of all team ratings is $\frac{1}{2}$ (check the examples above). If one team increases its rating, the rating of another team must decrease to keep this balance.
- Because it uses only win loss information, the Colley method can be used in a wider variety of situations, in particular in non-sporting examples.
- A draw between two teams is counted as neither a win nor a loss, but in the above system, it is counted as a game. If we remove draws from the data, the Colley ratings may change, hence it is important to decide what to do with draws before ranking the teams.

Extras: Time Weighting

Some games might be considered more important than others in a tournament and therefore should carry more weight in the rankings. For example, we can give games played later in the season a heavier weight than games played at the start of the season in a number of ways. We can do this in any way we please, however because of the amount of data involved, we should assign a weight that can be described as a function of time with a formula, so that we can program the formula into the computer. We will denote the weight we assign to a game between Team i and Team j as w_{ij} (To ensure that we can write formulas easily later, we will always write the weight with $i < j$). In fact to be more precise, when using time to weight games, we write $w_{ij}(t)$ for the weight to indicate that it depends on the time when the game was played.

The most commonly used functions for weighting are shown below, Linear, Logarithmic, Exponential and a Step Function. We measure time in days with t_0 denoting the the time of the season opener, or day one of the season.



$w_{ij}(t)$	weight given to a matchup between Team i and Team i at time t, written with $i < j$
t_0	time of season opener (e.g. day 1 of season occurs when $t_0 = 0$)
t_f	time of final game of season.
t_s	specific time during season to change step weighting.
t	time of game under consideration

The formulas for the above weighting functions are

Linear

$$w_{ij}(t) = \frac{t-t_0}{t_f-t_0}$$

Logarithmic

$$w_{ij}(t) = \ln\left(\frac{t-t_0}{t_f-t_0} + 1\right)$$

Exponential

$$w_{ij}(t) = e^{\frac{t-t_0}{t_f-t_0}}$$

Step

$$w_{ij}(t) = \begin{cases} 1 & \text{if } t \leq t_s \\ 2 & \text{if } \quad t > t_s \end{cases}$$

Example Let us look at the values of the weighting functions $w_{ij}(t)$ for the data below. This data is from the beginning of the season. We measure t in days. We set $t_0 = 0$ at the beginning of Jan 02. If we were going to use this data to predict what would happen on Jan 10, it would make sense to use $t_f = 8$ at the beginning of the day on Jan. 10. (If we were using this data when the last game of the season had been played on Mar 04 to predict what would happen in and $t_f = 61$ at the beginning of Mar 04.) We will round up the value of t for each game to a whole number, so $t = 1$ on Jan 02. Although early, we set $t_s = 5$ here just to demonstrate how to use the step function.
Fill in the values of $w_{ij}(t)$ that are missing (answers are given at the end of the lecture).

Date	Teams	Winner	t	$w_{ij}(t)$	$w_{ij}(t) = L$ Lin.	$w_{ij}(t) = \ln(L+1)$ Log.	$w_{ij}(t) = e^L$ Exp.	$w_{ij}(t)$ Step
Jan 02, 2013	BING vs HART	HART	1	$w_{49}(1)$	1/8	0.1178	1.1331	1
Jan 02, 2013	UVM vs UNH	UVM	1					
Jan 02, 2013	BU vs ME	ME	1					
Jan 02, 2013	ALBY vs UMBC	ALBY	1					
Jan 05, 2013	STON vs UNH	STON	4					
Jan 05, 2013	UVM vs ALBY	UVM	4					
Jan 05, 2013	BU vs HART	HART	4					
Jan 05, 2013	ME vs UMBC	ME	4					
Jan 07, 2013	BING vs ALBY	ALBY	6	$w_{59}(6)$	6/8	0.5596	2.117	2
Jan 08, 2013	UVM vs BU	BU	7					
Jan 09, 2013	BING vs STON	STON	8					
Jan 09, 2013	ME vs HART	HART	8					
Jan 09, 2013	UMBC vs UNH	UMBC	8					

Recall the numbers we assigned to each team which we use to fill in the i and j in $w_{ij}(t)$ for the games above:

i	Team i	Abbreviation
1	Stony Brook	STON
2	Vermont	UVM
3	Boston University	BU
4	Hartford	HART
5	Albany	ALBY
6	Maine	ME
7	Univ. Maryland, Bal. County	UMBC
8	New Hampshire	UNH
9	Binghampton	BING

Weighting The Colley Matrix

To introduce weights to the Colley ranking, we replace the number of matchups between Team i and Team j in the off diagonal entries of the Colley Matrix by the the sum of the weighted games between Team i and Team j. We also replace the number of wins and losses for Team i by the sum of the weighted wins and losses for Team i and we change the right hand vector b accordingly.

Colley's Method with time weights $w_{ij}(t)$, $i < j$

Colley's ratings with time weights $w_{ij}(t)$ are the solutions $\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$ of the linear system

$$C\mathbf{r} = \mathbf{b}$$

where
C is an $n \times n$ matrix called the Colley Matrix where

$$C_{ij} = \begin{cases} 2 + t_i & i = j \\ -\sum w_{ij}(t) & i < j \\ -\sum w_{ji}(t) & i > j \end{cases}$$

t_i sum of the weights of the games played by Team i, $t_i = \sum_{i<j} w_{ij}(t) + \sum_{i>j} w_{ji}(t)$.

$\mathbf{b}_{n\times 1}$ $n \times 1$ matrix on the right with $b_i = 1 + \frac{1}{2}(w_i - l_i)$

w_i total of the weighted wins accumulated by team i, $= \sum_{\text{i wins, } i<j} w_{ij}(t) + \sum_{\text{i wins, } i>j} w_{ji}(t)$.

l_i total of the weighted losses accumulated by team i, $= \sum_{\text{i loses, } i<j} w_{ij}(t) + \sum_{\text{i loses, } i>j} w_{ji}(t)$.

$\mathbf{r}_{n\times 1}$ general rating vector produced by the Colley system.
n number of teams in the conference = order of C.

Example Use the linear weights from the data in the previous example to set up the matrix system you get when you use the weighted Colley Method for our running example. Use your Mathematica file to solve the system of equations and derive the ratings.

Answers: Weighted Colley

Date	Teams	Winner	t	$w_{ij}(t)$	$w_{ij}(t) = L$ Lin.	$w_{ij}(t) = \ln(L+1)$ Log.	$w_{ij}(t) = e^L$ Exp.	$w_{ij}(t)$ Step
Jan 02, 2013	BING(9) vs HART(4)	HART(4)	1	$w_{49}(1)$	1/8	0.1178	1.1331	1
Jan 02, 2013	UVM(2) vs UNH(8)	UVM(2)	1	$w_{28}(1)$	1/8	0.1178	1.1331	1
Jan 02, 2013	BU(3) vs ME(6)	ME(6)	1	$w_{36}(1)$	1/8	0.1178	1.1331	1
Jan 02, 2013	ALBY(5) vs UMBC(7)	ALBY(5)	1	$w_{57}(1)$	1/8	0.1178	1.1331	1
Jan 05, 2013	STON(1) vs UNH	STON(1)	4	$w_{18}(4)$	4/8	0.4055	1.6487	1
Jan 05, 2013	UVM(2) vs ALBY(5)	UVM(2)	4	$w_{25}(4)$	4/8	0.4055	1.6487	1
Jan 05, 2013	BU(3) vs HART(4)	HART(4)	4	$w_{34}(4)$	4/8	0.4055	1.6487	1
Jan 05, 2013	ME(6) vs UMBC(7)	ME(6)	4	$w_{67}(4)$	4/8	0.4055	1.6487	1
Jan 07, 2013	BING(9) vs ALBY(5)	ALBY(5)	6	$w_{59}(6)$	6/8	0.5596	2.117	2
Jan 08, 2013	UVM(2) vs BU(3)	BU(3)	7	$w_{23}(7)$	7/8	0.6286	2.3989	2
Jan 09, 2013	BING(9) vs STON(1)	STON(1)	8	$w_{19}(8)$	1	0.6932	2.7183	2
Jan 09, 2013	ME(6) vs HART(4)	HART(4)	8	$w_{46}(8)$	1	0.6932	2.7183	2
Jan 09, 2013	UMBC(7) vs UNH(8)	UMBC(7)	8	$w_{78}(8)$	1	0.6932	2.7183	2

Weighted Colley using linear weights

$$
\begin{array}{lllll}
\text{STONY} & t_1 & = w_{18}(4) + w_{19}(8) & = 4/8 + 1 & = 12/8 \\
\text{UVM} & t_2 & = w_{28}(1) + w_{25}(4) + w_{23}(7) & = 1/8 + 4/8 + 7/8 & = 12/8 \\
\text{BU} & t_3 & = w_{36}(1) + w_{34}(4) + w_{23}(7) & = 1/8 + 4/8 + 7/8 & = 12/8 \\
\text{HART} & t_4 & = w_{49}(1) + w_{34}(4) + w_{48}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{ALBY} & t_5 & = w_{57}(1) + w_{25}(4) + w_{59}(6) & = 1/8 + 4/8 + 6/8 & = 11/8 \\
\text{ME} & t_6 & = w_{36}(1) + w_{67}(4) + w_{46}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{UMBC} & t_7 & = w_{57}(1) + w_{67}(4) + w_{78}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{UNH} & t_8 & = w_{28}(1) + w_{18}(4) + w_{78}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{BING} & t_9 & = w_{49}(1) + w_{59}(6) + w_{19}(8) & = 1/8 + 6/8 + 7/8 & = 15/8
\end{array}
$$

Wins

$$
\begin{array}{lllll}
\text{STONY} & w_1 & = w_{18}(4) + w_{19}(8) & = 4/8 + 1 & = 12/8 \\
\text{UVM} & w_2 & = w_{28}(1) + w_{25}(4) & = 1/8 + 4/8 & = 5/8 \\
\text{BU} & w_3 & = w_{23}(7) & = 7/8 & = 7/8 \\
\text{HART} & w_4 & = w_{49}(1) + w_{34}(4) + w_{48}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{ALBY} & w_5 & = w_{57}(1) + w_{59}(6) & = 1/8 + 6/8 & = 7/8 \\
\text{ME} & w_6 & = w_{36}(1) + w_{67}(4) & = 1/8 + 4/8 & = 5/8 \\
\text{UMBC} & w_7 & = w_{78}(8) & = 1 & = 1 \\
\text{UNH} & w_8 & & & = 0 \\
\text{BING} & w_9 & & & = 0
\end{array}
$$

Losses

$$
\begin{array}{lllll}
\text{STONY} & l_1 & & & = 0 \\
\text{UVM} & l_2 & = w_{23}(7) & = 7/8 & = 7/8 \\
\text{BU} & l_3 & = w_{36}(1) + w_{34}(4) & = 1/8 + 4/8 & = 5/8 \\
\text{HART} & l_4 & & & = 0 \\
\text{ALBY} & l_5 & = w_{25}(4) & = 4/8 & = 4/8 \\
\text{ME} & l_6 & = w_{46}(8) & = 1 & = 1 \\
\text{UMBC} & l_7 & = w_{57}(1) + w_{67}(4) & = 1/8 + 4/8 & = 5/8 \\
\text{UNH} & l_8 & = w_{28}(1) + w_{18}(4) + w_{78}(8) & = 1/8 + 4/8 + 1 & = 13/8 \\
\text{BING} & l_9 & = w_{49}(1) + w_{59}(6) + w_{19}(8) & = 1/8 + 6/8 + 7/8 & = 15/8
\end{array}
$$

We get $C =$

$$
\begin{pmatrix}
\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -1 \\
0 & \frac{7}{2} & -\frac{7}{8} & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{8} & 0 \\
0 & -\frac{7}{8} & \frac{7}{2} & -\frac{1}{2} & 0 & -\frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} & \frac{29}{8} & 0 & 0 & 0 & -1 & -\frac{1}{8} \\
0 & -\frac{1}{2} & 0 & 0 & \frac{27}{8} & 0 & -\frac{1}{8} & 0 & -\frac{3}{4} \\
0 & 0 & -\frac{1}{8} & 0 & 0 & \frac{29}{8} & -\frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{2} & \frac{29}{8} & -1 & 0 \\
-\frac{1}{2} & -\frac{1}{8} & 0 & -1 & 0 & 0 & -1 & \frac{29}{8} & 0 \\
-1 & 0 & 0 & -\frac{1}{8} & -\frac{3}{4} & 0 & 0 & 0 & \frac{31}{8}
\end{pmatrix}
$$

We get $\mathbf{b} =$

$$
\begin{pmatrix}
\frac{7}{4} \\ \frac{7}{8} \\ \frac{9}{8} \\ \frac{29}{16} \\ \frac{19}{16} \\ \frac{13}{16} \\ \frac{19}{16} \\ \frac{3}{16} \\ \frac{1}{16}
\end{pmatrix}
$$

Solving for $\mathbf{r}$ in $C\mathbf{r} = \mathbf{b}$, we get $\mathbf{r} =$

$$
\begin{pmatrix}
0.660519 \\ 0.48025 \\ 0.556631 \\ 0.726925 \\ 0.51123 \\ 0.316204 \\ 0.528323 \\ 0.505666 \\ 0.308982
\end{pmatrix}
$$

giving us the following ratings (from highest to lowest)

HART	0.726925
STONY	0.660519
BU	0.556631
UMBC	0.528323
ALBY	0.51123
UNH	0.505666
UVM	0.48025
ME	0.316204
BING	0.308982