1.1 Four Ways to Represent a Function

Functions are used to explain how one quantity depends on another in a unique and predictable way. Many everyday relationships between variables can be expressed as functions.

Example:

- Area A of a circle depends on its radius r: $A = \pi r^2$.
- Human population P depends on time t: $P(1950) = 2.56B$.
- Cost C of mailing a stamped letter (first-class mail, USPS, Dec. 2020) depends on its weight w.

Letters (Stamped)[1,4]

Weight Not Over (oz.)	
1	\$0.55
2	0.70
3	0.85
3.5	1.00

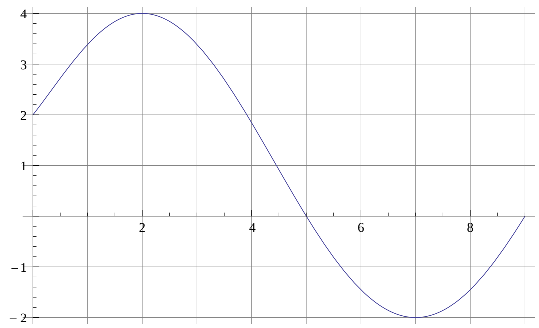
- Dorm in which a student resides depends on the student.
- Winning percentage $= \dfrac{\text{Games Won}}{\text{Games Played}}$.

A **function** is a rule that assigns to each element x in a set D a unique element $f(x)$ in a set E.
The **domain** of f is the set D.
The **range** of f is the set $\{f(x) \mid x \text{ in } D\} \subset E$.
The **graph** of f is the set of points $\{(x, f(x)) \mid x \text{ in } D\}$. The graph can also be described as the set of all points (x, y) on the Cartesian $(xy-)$ plane with $y = f(x)$.

A function can be thought of as an **input-output** machine: for each input x there is a output $f(x)$ uniquely determined by x.

Finding the Domain of a Function Given the graph of a function in the , the domain is the set of x values on the graph and the range is the set of y values on the graph.

Example: Find the domain and range of the function given by the graph.



When we describe a function by a formula, such as the one above, without specifying the domain, it is implicit that the domain is the set of all real values x which make sense in the formula. **Example:**

Find the domain of the functions:
(i) $f(x) = \sqrt{x-1}$.

(ii) $g(x) = \dfrac{1}{x-2}$.

Graphing a Function Two general observations will help us make a good start in graphing:

- The graph of an equation of the form $y = ax + b$, where a and b are real numbers is always a straight line, and two points determine such a line.
- The graph of a polynomial of the form $y = ax^2 = bx + c$ is always a parabola. One can get a good picture of the graph by completing the square, along with shifting and stretching techniques which we will discuss later.

Example: Sketch the graphs of the functions (a) $f(x) = 3 - x$,

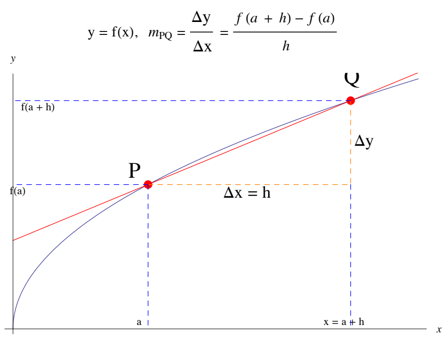
(b) $g(x) = x^2 - 4x + 1$.

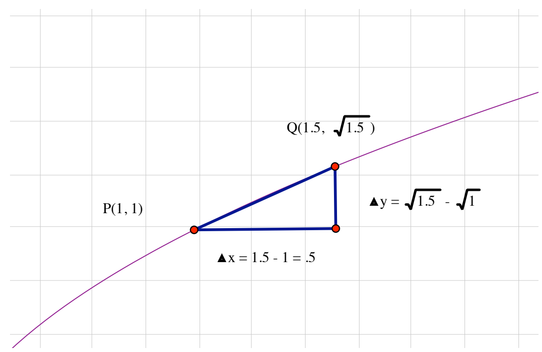
Average Rate of change of a Function

The average rate of change of a function $f(x)$ measures how much the values of the function (a.k.a. the output, a.k.a. the y-values on the graph) change per unit change in the values of x on a given interval. In general, for a function described by a formula $f(x)$, the average rate of change of the function will depend on the interval on which we compute it. For a given value of x, say $x = a$, if we change x by a small amount, say h, the average rate of change of the function f on the interval $[a, a+h]$ (h could be positive or negative here) is given by

$$\frac{\triangle f}{\triangle x} = \frac{\triangle y}{\triangle x} = \frac{f(a+h) - f(a)}{h}.$$

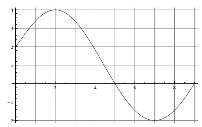
Here the notation $\triangle$ is shorthand for "The change in". If h is very small, this measure gives us an <u>estimate</u> of the slope of the graph of the function at the point $(a, f(a))$. The pictures below show a general graph of a function on the left, on the right the graph of $y = \sqrt{x}$ is shown and the picture refers to the calculation of $\dfrac{\triangle y}{\triangle x}$ at the point $x = 1$ with $h = 0.5$. In both cases the average rate of change $\dfrac{f(a+h) - f(a)}{h}$ give the slope of the line joining the points P and Q.





Example: Describe the average rate of change of the area of a circle with respect to a change in its radius by h units for any value of the radius r.

Ways to Represent a Function

- *Verbally*, as a rule. ex: $f(n) = n$-th digit of π
- *Numerically*, as a table of values. ex: $C(w) = \begin{cases} 0.55 & 0 < w \leq 1 \\ 0.70 & 1 < w \leq 2 \\ 0.85 & 2 < w \leq 3 \\ 1.00 & 3 < w \leq 3.5 \end{cases}$
- *Visually*, as a graph. ex: 
- *Algebraically*, as an explicit formula using a variable x. ex: $f(x) = x^2 - 4x + 1$.

Checking if two functions are the same. In order for two functions to be the same, their domains must be the same and the functions must produce exactly the same output for each input value. Since there are a number of ways to represent functions, and a number of ways to manipulate algebraic formulas, functions that look quite different sometimes turn out to be the same.

Example: Determine whether the given functions are the same function.

- $f(x) = \dfrac{x^2 - x}{x - 1}, \quad g(x) = x$

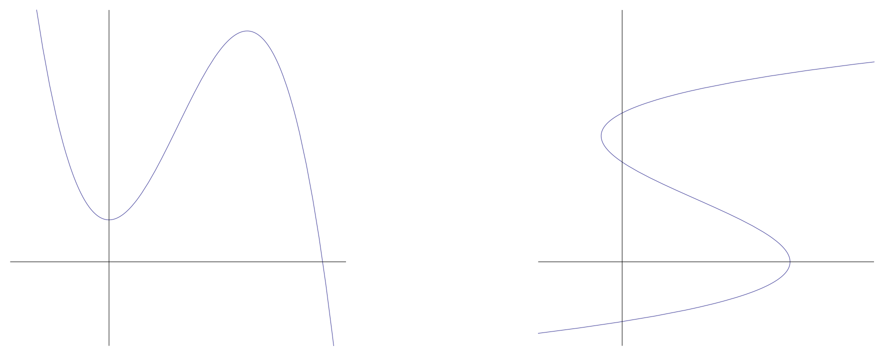
- $f(x) = \dfrac{1}{\sqrt{x+1} - \sqrt{x}} \quad g(x) = \sqrt{x+1} + \sqrt{x}$

- $f(x) = x^2$, $g(u) = u^2$

Vertical Line Test: Every equation relating two variables x and y has a graph on the xy-plane, namely the set of all pairs (x, y) which satisfy the equation.

Example: The graph of the equation $x^2 + y^2 = 25$ is the graph of a circle of radius 5 centered at the point $(0, 0)$.

A curve is the graph of a function $y = f(x)$ if and only if no vertical line intersects the curve more than once.
Example: Which of the following graphs is the graph of a function?



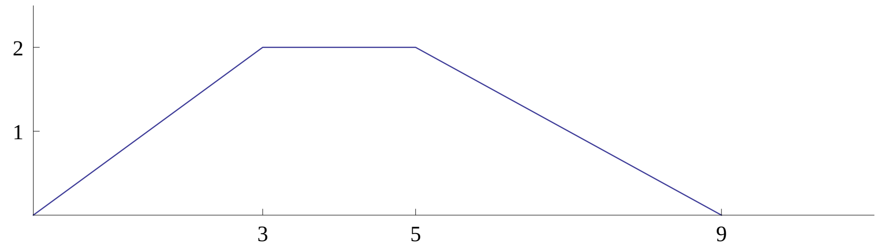
Piecewise Defined Functions Some functions cannot be described by a single formula and may have different formulas on different sections of the real line.
The domain of such a piecewise defined function is the set of all numbers for which the function is defined.
Example: $f(x) = \begin{cases} x+1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$
Graph the above function and find its domain and range.

The Absolute Value Function

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Graph the absolute value function and find its domain.

Example: Find a formula for $f(x)$ defined by the graph.



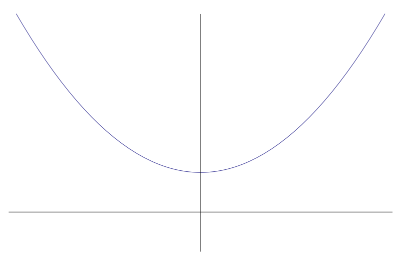
Symmetry

$f(x)$ is an **even** function if $f(-x) = f(x)$. The graph of an even function for $x < 0$ is a reflection of the graph for $x > 0$ through the y-axis.

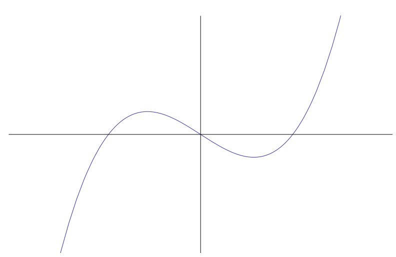
$f(x)$ is an **odd** function if $f(-x) = -f(x)$. The graph of an odd function for $x < 0$ is a reflection of the graph for $x > 0$ through the origin $(0, 0)$.

Example: Determine if the following function is even or odd and verify that the graph has the predicted symmetry.

$f(x) = x^2 + 1$.



Example: $g(x) = x^3 - x$.



$f(x)$ is **increasing** on an interval if for any two numbers in that interval with $x_1 < x_2$ we have $f(x_1) < f(x_2)$.
$f(x)$ is **decreasing** on an interval if for any two numbers in that interval with $x_1 < x_2$ we have $f(x_1) > f(x_2)$.

Example: Is the function $f(x) = \sqrt{1 - x^2}$ even or odd?
Sketch the graph of the function.
Determine the intervals on which $f(x)$ is increasing and those on which it is decreasing using the graph.