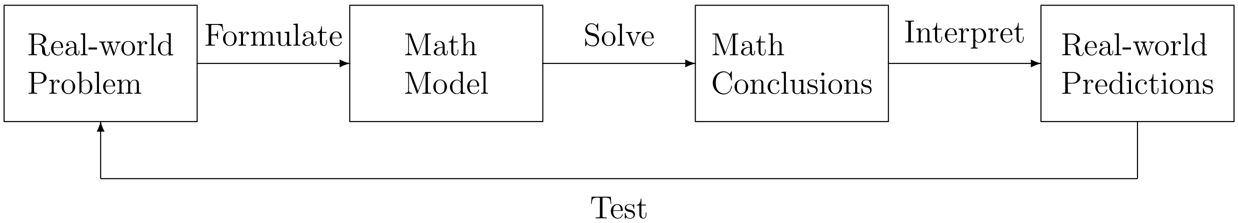
1.2 Mathematical Models: A Catalog of Essential Functions

The process of modeling real-world problems has several components.

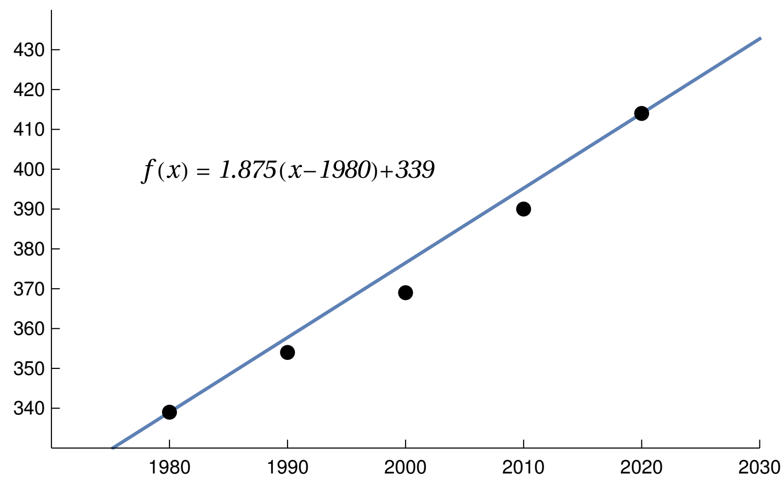


A mathematical model is an *idealized* representation of a physical situation. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. We now discuss the behavior and graphs of some types of functions that can be used to model relationships observed in the real world.

Linear Models

Example: The CO_2 levels in the atmosphere observed during the past 4 decades are given in the table. Find a function that models these CO_2 levels and use it to predict the amount of CO_2 in the atmosphere in 2030.

year	ppm
1980	339
1990	354
2000	369
2010	390
2020	414



The points seem to line up. We can approximate the slope m using the first and last points, and then use the point-slope formula $y = m(x - x_1) + y_1$ to arrive at a linear function that models the CO_2 levels:

$$m = \frac{414 - 339}{2020 - 1980} = \frac{75}{40} = 1.875, \qquad f(x) = 1.875(x - 1980) + 339$$

The slope $m = 1.875$ represents the average increase in CO_2 per year. In 2030, the model predicts the level will be $f(2030) = 1.875(50) + 339 = 432.75$ ppm.

Polynomials

A polynomial is a function of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. The numbers $a_0, \ldots, a_n$ are called the **coefficients** of $p(x)$ and the positive integer n is called the degree of $p(a)$ (assuming $a_n \neq 0$).

Example: Consider $p(x) = 3x^5 - \sqrt{2}\,x + \pi$. The degree is 5. The coefficients are $a_0 = \pi$, $a_1 = -\sqrt{2}$, $a_2 = a_3 = a_4 = 0$, and $a_5 = 3$.

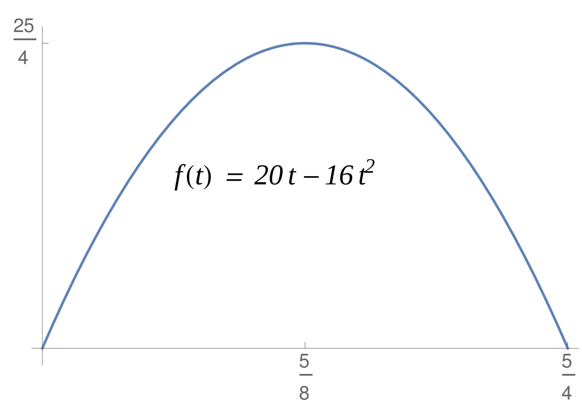
Special names are given to low-degree polynomials:
degree 1: $p(x) = mx + b$, **linear**, graphs are lines

degree 2: $p(x) = ax^2 + bx + c$, **quadratic**, graphs are parabolas

degree 3: $p(x) = ax^3 + bx^2 + cx + d$, **cubic**, graphs have at most two "bumps."

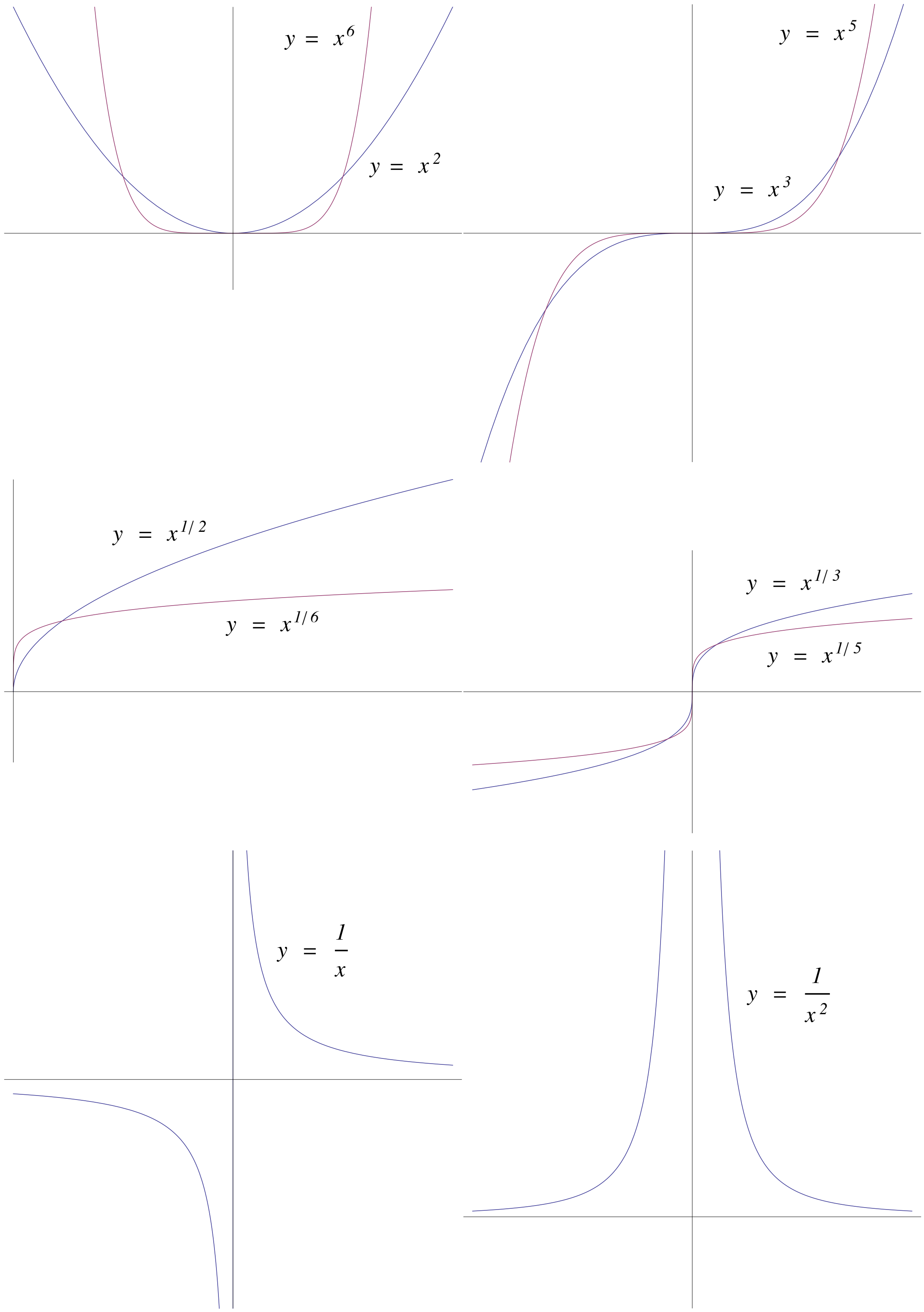
Example: Find a cubic polynomial $p(x)$ satisfying $p(0) = 0$, $p(1) = 0$, and $p(2) = 0$ and $p(-1) = -6$.

Example: If a ball is thrown upward with an initial velocity of 20 ft/sec, its height after t seconds is given by $f(t) = 20t - 16t^2$ ft. Find the maximum height of the ball and when it hits the ground.



Power Functions

A power function has the form $f(x) = x^a$ for some constant a. For example $f(x) = x^6$, or $f(x) = x^{1/3}$, or $f(x) = x^{-2} = 1/x^2$.



Rational Functions

A rational function has the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Examples: $f(x) = \dfrac{2x^3 - x + 12}{x^2 + x + 1}, \quad g(x) = \dfrac{1}{x^2+1}$

Algebraic Functions

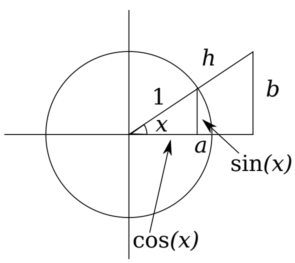
An algebraic function is one that is constructed from any combination of algebraic operations, including taking powers and roots.

Examples: $f(x) = \dfrac{x^2 + \sqrt[3]{x^2} + 1}{x+1}, \quad g(x) = \dfrac{x^4 - 16x}{1 + \sqrt{x}}, \quad m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}},$

In relativity, the mass m of an object is an algebraic function of its velocity v relative to the observer (m_0 is the rest mass and c is the speed of light).

Trigonometric Functions

The trigonometric functions sine, $\sin(x)$, and cosine, $\cos(x)$, are defined by the coordinates of a point $P(x) = (\cos(x), \sin(x))$ on a unit circle at an angle of x from the horizontal axis. By similar triangles, these functions also give the ratios of sides of a right triangle.



$$\sin(x) = \frac{b}{h}, \quad \cos(x) = \frac{a}{h}, \quad \tan(x) = \frac{b}{a}$$

We extend the definition of $\sin(x)$ and $\cos(x)$ to all real numbers x by defining

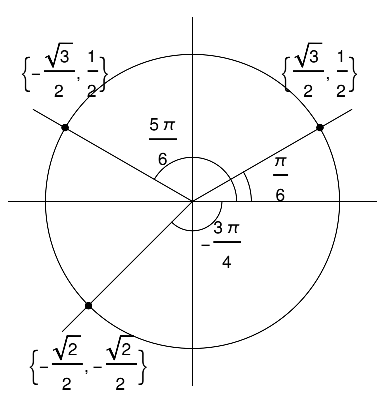
$$\sin(x + 2\pi) = \sin(x), \quad \cos(x + 2\pi) = \cos(x)$$

The values of $\sin(x)$ and $\cos(x)$ always lie between -1 and 1. The other trigonometric functions are defined in terms of $\sin(x)$ and $\cos(x)$:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}$$

They are also defined for all real numbers x, except where the denominator is 0. The following special values of $\sin(x)$ and $\cos(x)$ for angles in the first quadrant should be remembered. The values for other quadrants may be easily deduced from these. For example, $\cos(5\pi/6) = -\sqrt{3}/2$, $\sin(5\pi/2) = 1/2$, or $\cos(-3\pi/4) = -\sqrt{2}/2$, $\sin(-3\pi/4) = -\sqrt{2}/2$.

x	$\sin(x)$	$\cos(x)$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

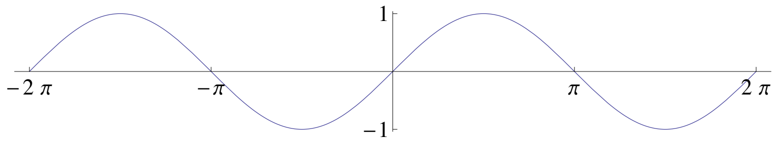


From the definition we obtain the identities

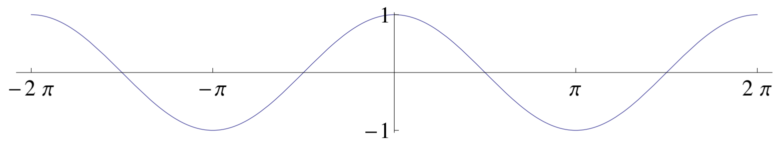
$$\sin^2(x) + \cos^2(x) = 1, \quad \sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x)$$

and by dividing the first identity by $\cos^2(x)$ we obtain the identity $\tan^2(x) + 1 = \sec^2(x)$.

$\sin(x)$:

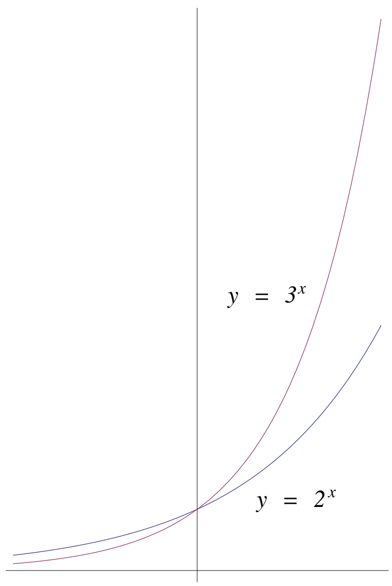


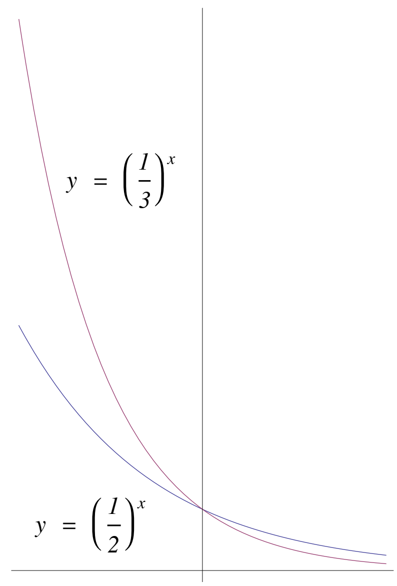
$\cos(x)$:



Exponential Functions

An exponential function has the form $f(x) = a^x$ for some positive constant $a > 0$. An exponential function with $a > 1$ rapidly increases, while if $0 < a < 1$, the function rapidly decreases.

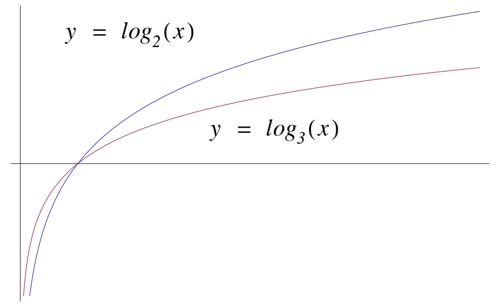




Example: A population of bacteria starts with 1000 cells and doubles every 3 hours. Find a function that models the bacteria growth at any time t.

Logarithmic Functions

Logarithmic functions are defined in terms of exponential functions: For any positive constant $a > 0$: $y = \log_a(x) \Longleftrightarrow a^y = x$.



Example:

$\log_2(64) = 6$ because $2^6 = 64$ $\quad$ $\log_4(64) = 3$ because $4^3 = 64$

$\log_8(64) = 2$ because $8^2 = 64$ $\quad$ $\log_{16}(64) = 3/2$ because $(16)^{3/2} = 4^3 = 64$

$\log_{64}(64) = 1$ because $64^1 = 64$

Example: Classify the functions.

(a) $f(x) = \pi^x$:

(b) $f(x) = x^\pi$:

(c) $f(x) = x^2(2 - x^3)$:

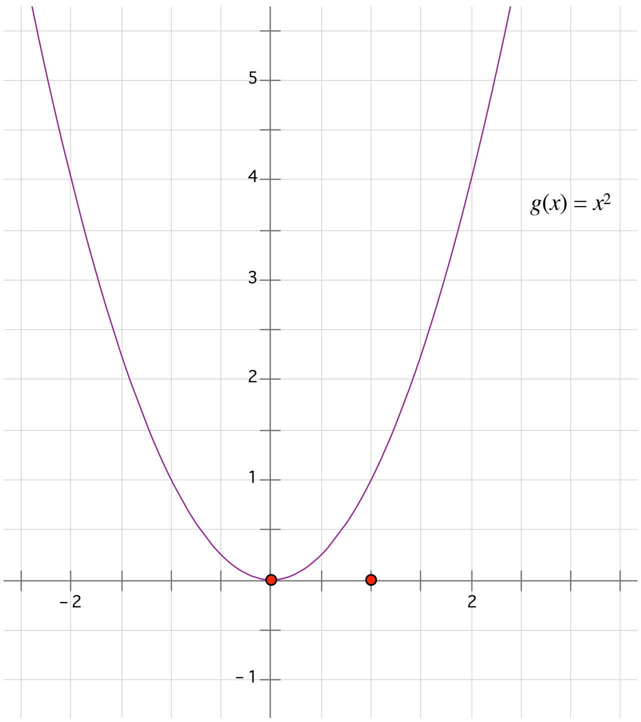
(d) $f(t) = \tan(t) - \cos(t)$:

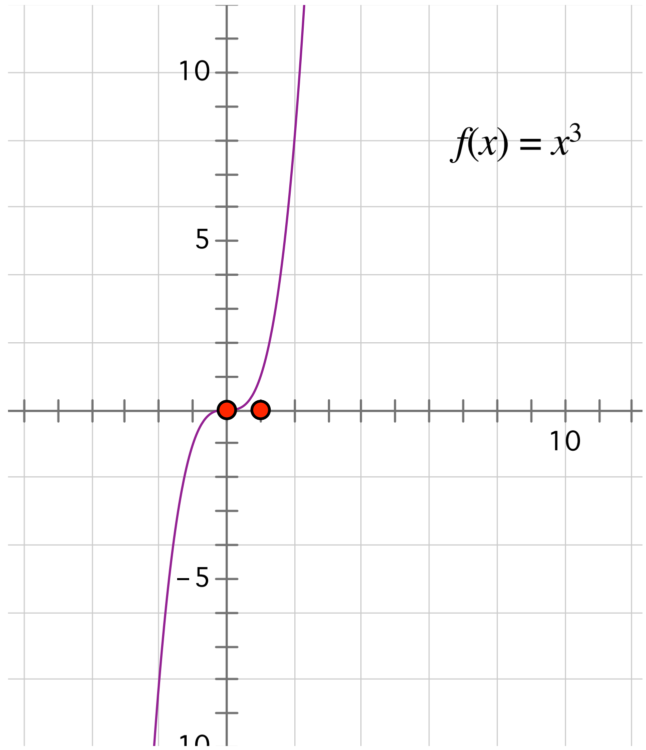
(e) $f(s) = \dfrac{s}{1+s}$:

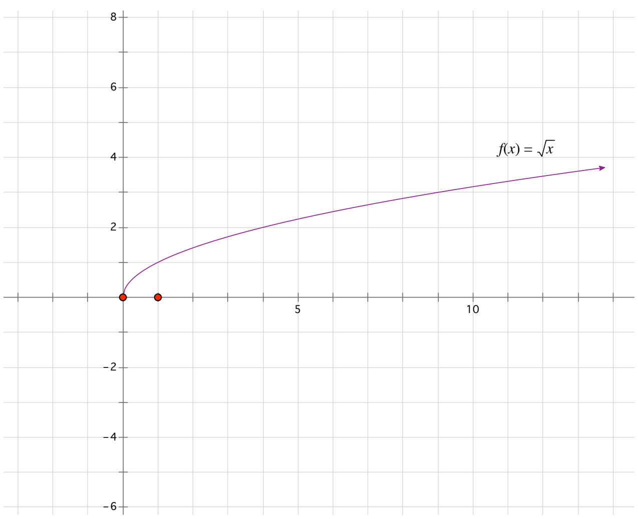
(f) $f(x) = \dfrac{x^3 - 1}{1 + \sqrt[3]{x}}$:

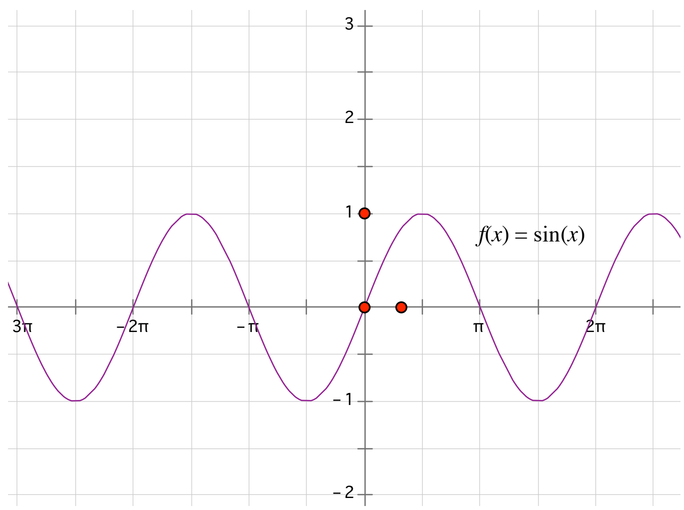
Appendix

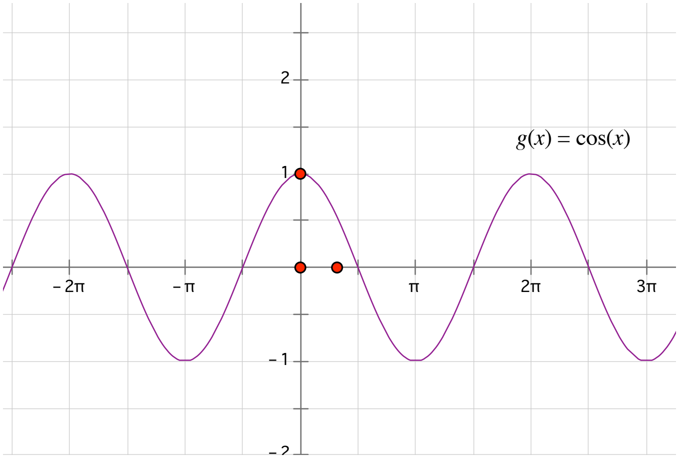
Catalogue of Graphs

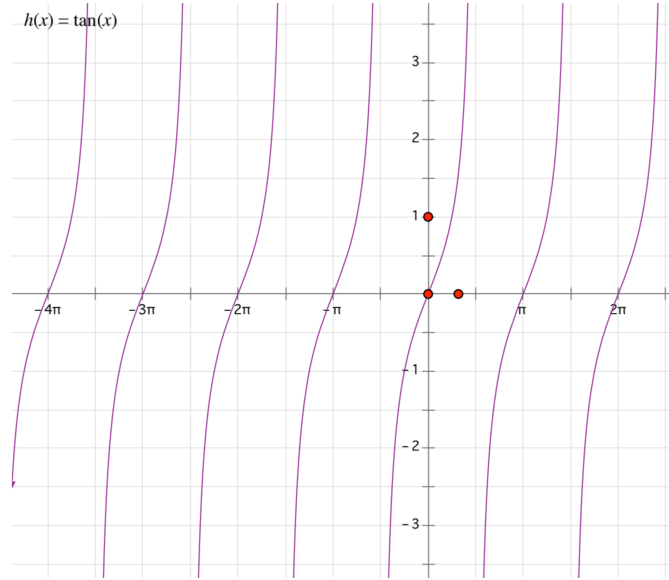


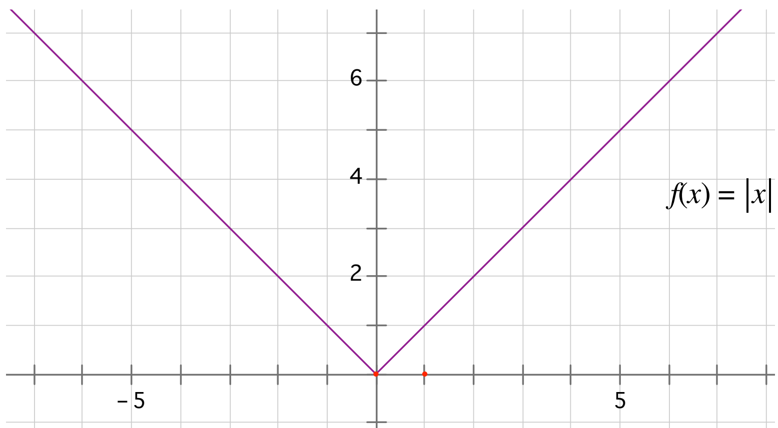


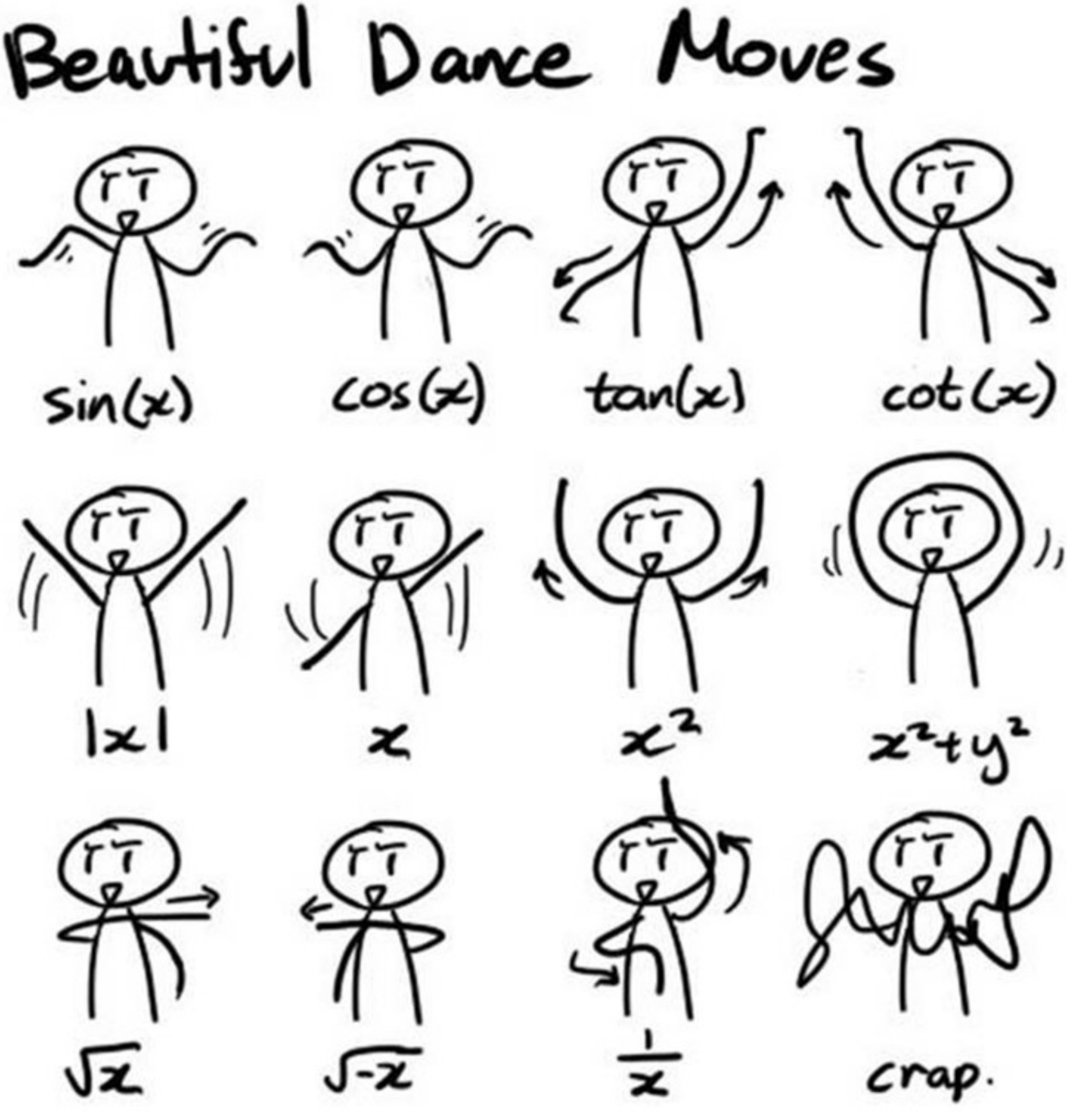





$g(x) = \cos(x)$
2
1
− 1
− 2
− 2π
− π
π
2π
3π


$h(x) = \tan(x)$
3
2
1
− 1
− 2
− 3
− 4π
− 3π
− 2π
− π
π
2π


$f(x) = |x|$
6
4
2
− 5
5


Beautiful Dance Moves
sin(x)
cos(x)
tan(x)
cot(x)
|x|
x
x²
x²+y²
√x
√-x
1/x
crap.