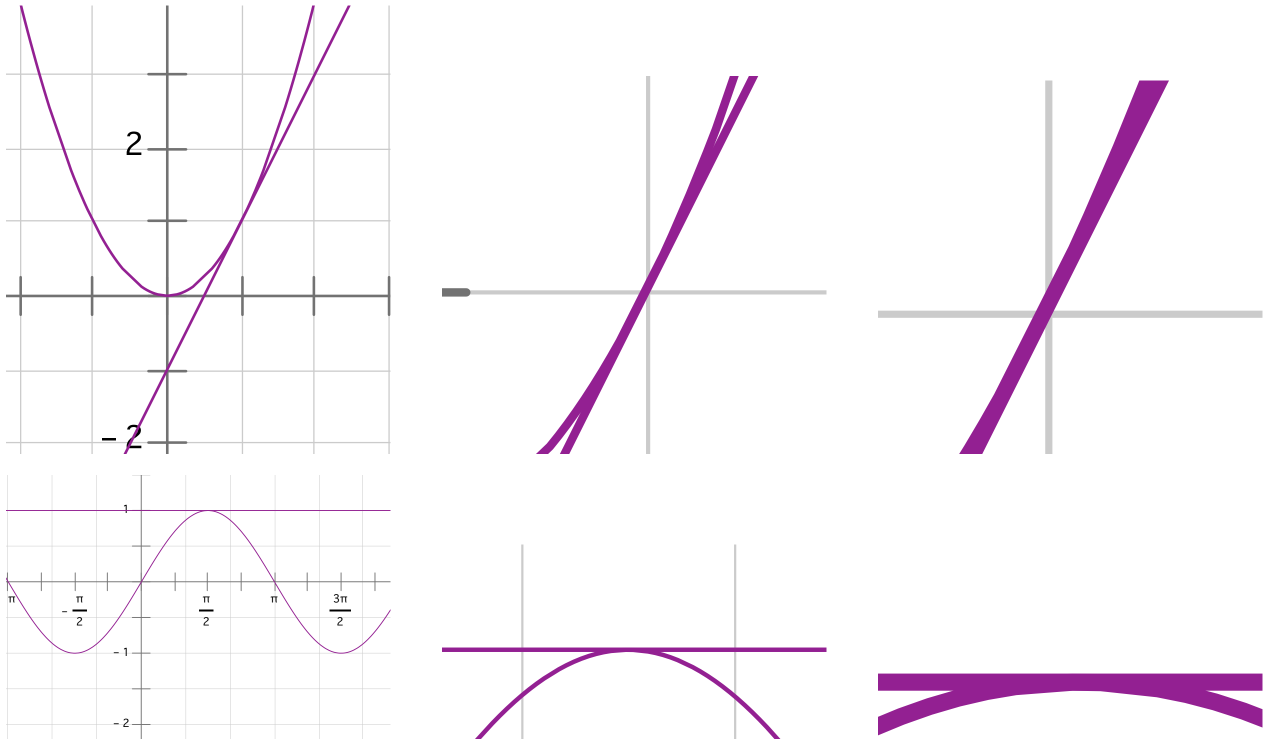
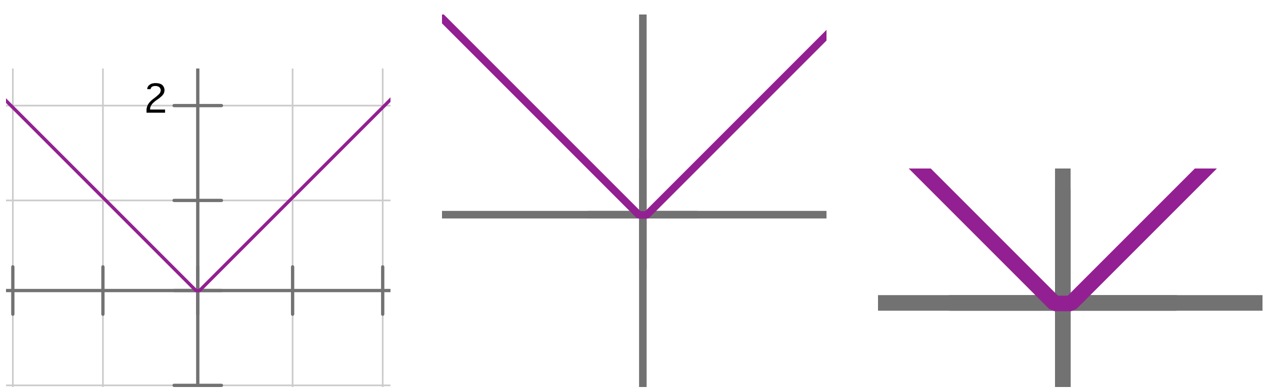
Lecture 14 :Linear Approximations and Differentials

Consider a point on a smooth curve $y = f(x)$, say $P = (a, f(a))$, If we draw a tangent line to the curve at the point P, we can see from the pictures below that as we zoom in towards the point P, the path of the curve is very close to that of the tangent line. If we zoom in far enough the curve looks almost linear near the point P. (See below the tangent to the curve $y = x^2$ at the point $(1, 1)$ and the tangent to the curve $y = \sin x$ at the point $(\frac{\pi}{2}, 1)$.)



On the other hand, if we pick a sharp point on a curve, the curve does not look linear near the point as we zoom in. See the graph of $y = |x|$ below at the point $(0, 0)$.



The difference between the nature of sharp points and points where the curve is smooth is differentiability. If a curve is differentiable on an interval containing the point $x = a$ (smooth at $x = a$), the points on the curve, with x values close to a, are very close to the points on the tangent line to the curve at P. The **equation of the tangent line to the curve** $y = f(x)$ **at** $x = a$ is given by,

$$\boxed{L(x) = f'(a)(x - a) + f(a).}$$

We can conclude that **if f is differentiable in an interval containing** a, then

$$\boxed{f(x) \approx L(x) = f(a) + f'(a)(x-a).}$$

This is called the **linear approximation** or **Tangent Line Approximation** to $f(x)$ at $x = a$. The linear function, whose graph is the tangent line to the curve $y = f(x)$ at $x = a$ is called the **Linearization** of f at a.

Example (a) Find the linearization of the function $f(x) = \sqrt[3]{x}$ at $a = 27$.

$f(x)$	$f'(x)$	a	$f(a)$	$f'(a)$

$L(x) = f(a) + f'(a)(x-a) =$

(b) Use the linearization above to approximate the numbers $\sqrt[3]{27.01}$ and $\sqrt[3]{26.99}$.

(c) We can get an approximation for $\sqrt[3]{x}$ from the linearization of the function, $L(x)$, above, for any x in the interval $26 \leq x \leq 28$. We can see, from the table below, that the closer the value of x gets to 27, the better the approximation to the actual value of $\sqrt[3]{x}$.

$f(x)$	x	From $L(x)$	Actual Value $\sqrt[3]{x}$
$\sqrt[3]{26.5}$	26.5	2.9814815	2.9813650
$\sqrt[3]{26.9}$	26.9	2.996296	2.996292
$\sqrt[3]{26.99}$	26.99	2.9996296	2.9996296
$\sqrt[3]{27}$	27	3	3
$\sqrt[3]{27.01}$	27.01	3.0003704	3.0003703
$\sqrt[3]{27.1}$	27.1	3.0037037	3.0036991
$\sqrt[3]{27.5}$	27.5	3.0185185	3.0184054

Error of approximation In fact by zooming in on the graph of $f(x) = \sqrt[3]{x}$, you will see that

$$|\sqrt[3]{x} - L(x)| < 0.001 \quad \text{or} \quad -0.001 < \sqrt[3]{x} - L(x) < 0.001$$

when x is in the interval $26.5 \leq x \leq 27.5$.

Such bounds on the error are useful when using approximations. We will be able to derive such estimates later when we study Newton's method.

Example (a) Find the linearization of the function $f(x) = \sqrt{x+9}$ at $a = 7$.

$f(x)$	$f'(x)$	a	$f(a)$	$f'(a)$

$L(x) =$

(b) Use the linearization above to approximate the numbers $\sqrt{16.03}$ and $\sqrt{15.98}$.

Example (commonly used linearizations) (a) Find the Linearization of the functions $\sin\theta$ and $\cos\theta$ at $\theta = 0$.

$f(\theta)$	$f'(\theta)$	a	$f(a)$	$f'(a)$

$\cos(\theta) \approx$

$\sin(\theta) \approx$

(b) Estimate the value of $\sin(\frac{\pi}{100})$, $\cos\frac{\pi}{95}$ and $\sin 2^o = \sin\frac{\pi}{90}$.

New Notation : Δy.

Suppose f is differentiable on an interval containing the point a. Linear approximation says that the function f can be approximated by

$$f(x) \approx f(a) + f'(a)(x - a)$$

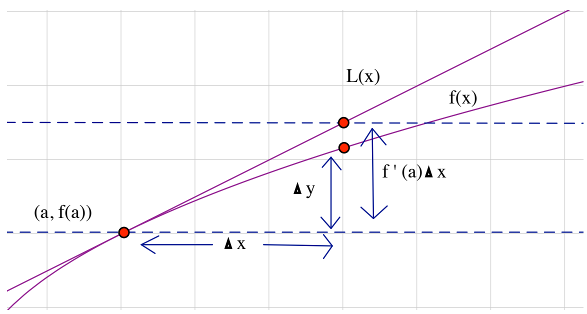
where a is fixed and x is a point (in the interval) nearby. This can gives us the following **approximation for the change in function values**, when we have a small change in the value of x:

$$\boxed{f(x) - f(a) \approx f'(a)(x - a).}$$

As before, we use Δx to denote a small change in x values. In this case $\Delta x = x - a$. and $\boxed{\Delta y = f(x) - f(a)}$ to denote the corresponding change in the values of y or $f(x)$. This gives us:

$$\boxed{\Delta y \approx f'(a)\Delta x}$$

where Δy denotes the change in the value of f between two points a and $a + \Delta x$.



Example Approximate the change in the surface area of a spherical hot air balloon when the radius changes from 4 to 3.9 meters. (The surface area of a sphere of radius r is given by $S = 4\pi r^2$.)

$$\Delta S \approx S'(a)\Delta r$$

$$S(r) = 4\pi r^2, \qquad S'(r) = 8\pi r.$$

$$\Delta S \approx S'(4)\Delta r = 32\pi(-0.1) = -3.2\pi.$$

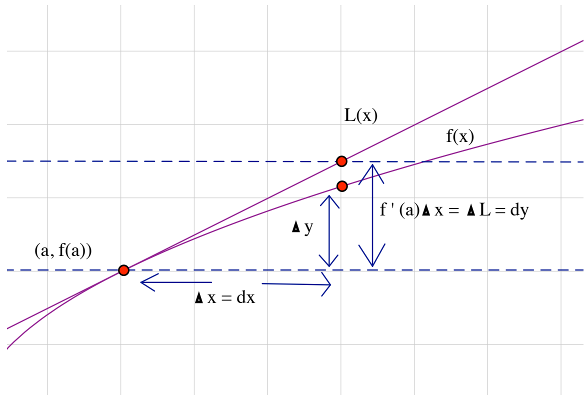
Differentials, dy

We also use the notation of **differentials** to denote changes in $L(x)$, the linear approximation to $f(x)$ near a.

- The change in the function values on the curve $y = f(x)$ as x changes from a to $a + \Delta x$ is denoted by Δy as before. ($\Delta y = f(a + \Delta x) - f(a)$).
- The differential dx is defined as the change in x, ($dx = \Delta x$.)
- The differential dy is defined as the change in the values of the linear approximation $L(x)$ as x changes from a to $a + \Delta x$; $dy = \Delta L = L(a + \Delta x) - L(a) = f(a) + f'(a)\Delta x - f(a) = f'(a)\Delta x$. Therefore

$$\boxed{dy = f'(a)\Delta x \quad \text{in terms of } dx\text{:} \quad dy = f'(a)dx.}$$

The differential dy is a dependent variable, depending on the independent variable dx. Check out the difference between dy and Δy on the graph below.



We can rephrase our results on linear approximation as

$$\boxed{dy \approx \Delta y}$$

Example Compare the values of Δy and dy if $y = 3x^4 + 2x + 1$ and x changes from 2 to 2.04.

$a = 2$

$\Delta y = f(a + \Delta a) - f(a) = f(2.04) - f(2) = 57.0367 - 53 = \boxed{4.0367 = \Delta y}$

$\Delta x = .04$

$f(x) = 3x^4 + 2x + 1$ $\quad$ $f'(x) = 12x^3 + 2$

$dy = f'(a)dx = (12(4) + 2)(.04) = 98(.04) = \boxed{3.92 = dy}$

Example (a) Find the differential for the function $y = 3\cos^2 x$.

$dy = f'(x)dx = (6\cos x)(-\sin x)dx = -6\cos x \sin x dx$

(b) Use the differential to approximate the change in the values of the function $f(x) = 3\cos^2 x$ when we have a small change in the value of x, dx at $x = \frac{\pi}{4}$.

$$f(x + \Delta x) - f(x) \approx dy = f'(\frac{\pi}{4})dx = -6\cos(\frac{\pi}{4})\sin(\frac{\pi}{4})dx = -6\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}dx = -3dx.$$

(c) Use differentials to estimate $3\cos^2(44^o) = 3\cos^2(\frac{\pi}{4} - \frac{\pi}{180})$.

$$3\cos^2(44^o) = 3\cos^2\left(\frac{\pi}{4} - \frac{\pi}{180}\right) \approx 3\cos^2\left(\frac{\pi}{4}\right) - 3\left(-\frac{\pi}{180}\right) = \frac{3}{2} + \frac{\pi}{60}.$$

Estimating Error

Note that Δy measures the resulting error in our value for the variable y if we make a mistake in our calculations of the variable x of size Δx, $\Delta y = f(x + \Delta x) - f(x)$. We saw above that $dy \approx \Delta y$ and we can use differentials to approximate the maximum error in our calculations for y when we have some bound on our error for the variable x.

Example The radius of a spherical hot air balloon was estimated to be 4 meters with a possible error of at most 0.5 meters. What is the maximum error you can make in calculating the surface area of the balloon using the estimate of 4 meters?

$$S = 4\pi r^2 \ m^2$$

We need bounds for ΔS here, but we will instead use the linear approximation $dS \approx \Delta S$ to approximate the error.

$$dS = 8\pi r dr$$

When $r = 4$,

$$dS = 32\pi dr$$

If $-0.5 \leq dr \leq 0.5$, then

$$-0.5(32\pi) \leq dS \leq 0.5(32\pi)$$

or

$$-50.26 \leq dS \leq 50.26.$$

We can interpret this result as saying that if our estimate of 4 meters for the radius of the balloon is off by at most 0.5 meters, then our estimate of the surface area of the balloon is off by (approximately) at most 50.26 meters squared in absolute value.

We can also find bounds for the **Relative Error** $= \frac{\Delta S}{S}$ and the **Percentage Error** $= \frac{\Delta S}{S} \cdot 100\%$.

$$\text{relative error} = \frac{\Delta S}{S} \approx \frac{dS}{S}$$

When $r = 4$, $S = 4\pi(16) = 201.06$. From our calculations above the relative error is at most $\frac{50.26}{201.06} = .25$. The ma percentage error is at most 25%.

Commonly Used Linear Approximations

Note that if $x \approx 0$, we get the following approximations for some commonly used functions using Linear approximation:

1. $\sin x \approx x$ if $x \approx 0$
2. $\cos x \approx 1$ if $x \approx 0$
3. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ if $x \approx 0$
4. $(1+x)^r \approx 1 + rx$ if $x \approx 0$.

Recall for $x \approx 0$, $f(x) \approx f(0) + f'(0)x$.

The above results come from the following table which you should verify:

$f(x)$	$f(0)$	$f'(x)$	$f'(0)$
$\sin x$	0	$\cos x$	1
$\cos x$	1	$-\sin x$	0
$(1+x)^r$	1	$r(1+x)^{r-1}$	r