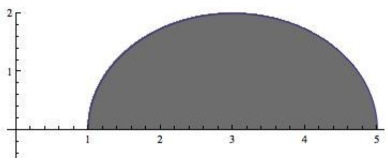
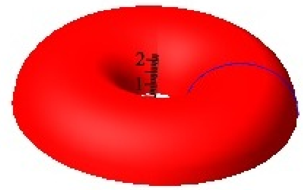
Volumes by Cylindrical shells

Example Consider the solid generated by rotating the region between the curve $y = \sqrt{4-(x-3)^2}$ and the line $y = 0$ (shown on the left below) about the y **axis.**
The solid looks like the half doughnut shown on the right below below:





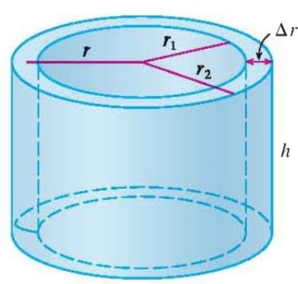
We see that in the example above it is difficult to apply the methods of the previous section to find the volume. Below we give a method, The shell method, which applies much more readily to this situation.

Volume of a shell A shell is a hollow cylinder such as the one shown below. The outer radius of the shell shown below is r_2 and the inner radius is r_1. Since all cross sections of the shell are the same, the volume of the shell is the area of the base times the height, h. The base has area

$$\pi r_2^2 - \pi r_1^2 = \pi(r_2 - r_1)(r_2 + r_1) = 2\pi \frac{(r_2 + r_1)}{2}(r_2 - r_1) = 2\pi r \Delta r,$$

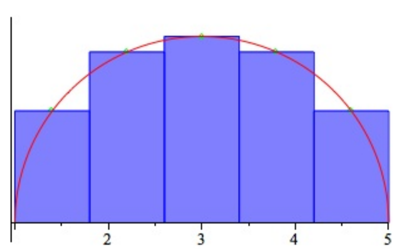
where r is the midpoint of the interval $[r_1, r_2]$ and $\Delta r = (r_2 - r_1)$. Therefore we have the volume of the shell shown below is

$$\boxed{V = 2\pi r h \Delta r, \quad \text{where} \quad r = \frac{r_1 + r_2}{2}.}$$



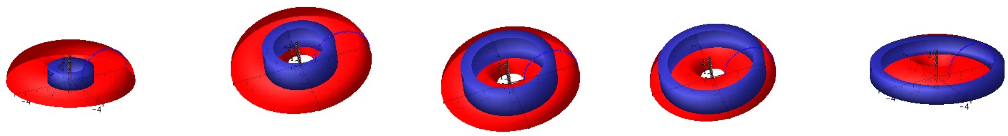
Let us consider the example given above :
Find the volume of the solid generated by rotating the region between the curve $y = \sqrt{4-(x-3)^2}$ and the line $y = 0$ about the y axis.

The area of the region under the curve $y = \sqrt{4-(x-3)^2}$ and the line $y = 0$ can be approximated using a midpoint approximation with $n = 5$ approximating rectangles as shown below.

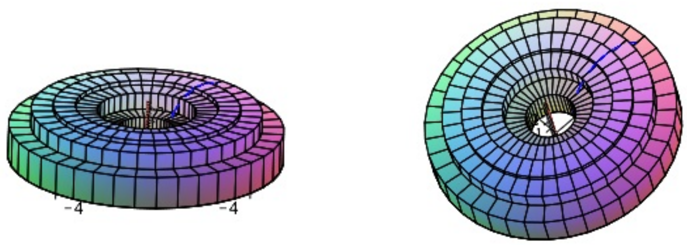


Recall that the base of a rectangle is a subinterval $[x_{i-1}, x_i]$ of length $\Delta x = \frac{5-1}{5} = 4/5$. When we rotate each approximating rectangle about the y axis, it generates a solid which is a shell with inner radius x_{i-1}, outer radius x_i and height $f(x_i^{mid}) = f(\frac{x_{i-1}+x_i}{2})$, where $x_i^{mid} = \frac{x_{i-1}+x_i}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$. The volume of this shell (generated by the approximating rectangle above the subinterval $[x_{i-1}, x_i]$) is

$$V_i = 2\pi \frac{x_{i-1}+x_i}{2} \Delta x \cdot h = 2\pi \frac{x_{i-1}+x_i}{2} f(\frac{x_{i-1}+x_i}{2}) \Delta x.$$



We can approximate the volume of the solid above by adding together the volumes of these shells:



We get

$$V \approx \sum_{i=1}^{5} 2\pi \frac{(x_i + x_{i-1})}{2} (x_i - x_{i-1}) f\left(\frac{x_i + x_{i-1}}{2}\right) = \sum_{i=1}^{5} 2\pi x_i^{mid} f(x_i^{mid}) \Delta x,$$

where x_i^{mid} is the midpoint of the interval $[x_{i-1}, x_i]$ and $\Delta x = x_i - x_{i-1}$. Following the same process with n approximating rectangles and n approximating shells, we get

$$V \approx \sum_{i=1}^{n} 2\pi x_i^{mid} f(x_i^{mid}) \Delta x$$

and using limits we get

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^{mid} f(x_i^{mid}) \Delta x = \int_1^5 2\pi x f(x) dx = 2\pi \int_1^5 x\sqrt{4-(x-3)^2} dx$$

$=$

- In general, let R be the region between the curve $y = f(x) \geq 0$ and the lines $y = 0$ (x axis), $x = a$ and $x = b$, where $0 \leq a < b$. Let S be the solid generated by revolving R around the y axis. The volume of S is given by

$$V = \int_a^b 2\pi x f(x) dx$$

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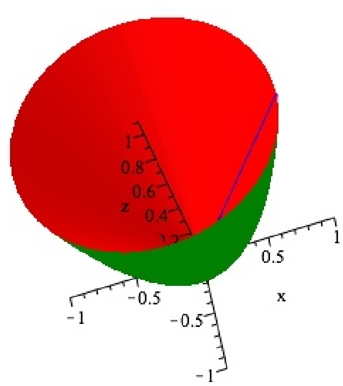
Example Let R denote the region enclosed by the curve $y = x^3$ and the lines $x = 0$, $x = 1$ and the x axis, $y = 0$. Find the volume of the solid generated by rotating R about the y axis.

- Suppose $f(x) \geq g(x) \geq 0$ on the interval $[a, b]$, where $0 \leq a < b$. Let R be the region enclosed by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$. Let S denote the solid obtained by rotating the region R about the y axis. The Volume of S is given by

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx.$$

Example Let R denote the region between the curves $y = x$ and $y = x^2$. Use the shell method to find the volume of the solid generated when R is rotated around the y axis.



Revolving around the line $x = c$.

- Let R be the region between the curve $y = f(x) \geq 0$ and the x axis, between the lines $x = a$ and $x = b$. Let S be the solid generated by revolving R around the line $x = c$ (parallel to the y axis) where $c \leq a < b$. The volume of S is given by

$$V = \int_a^b 2\pi(x - c)f(x)dx.$$

For $c \geq b > a$ we use the formula

$$V = \int_a^b 2\pi(c - x)f(x)dx.$$

Example Let R be the region between the curve $y = x^2$ and the lines $x = 0$(y-axis), $x = 1$, $y = 0$. Let S be the solid generated by revolving R around the line $x = 2$. Find the volume of S.

Shell method when revolving a function of y about the x axis.

- Let R be the region bounded by the curve $x = f(y)$ and the lines $x = 0$(y-axis), $y = c$ and $y = d$, where $0 \leq c < d$. Let S denote the solid generated by revolving the region R around the x axis. The volume S is given by:
$$V = \int_c^d 2\pi y f(y) dy.$$

Example Let R be the region bounded by the curve $x = y^3$ and the lines $y = 1$, $y = 2$ and $x = 0$(the y-axis). Find the volume of the solid generated by rotating R about the x axis.

Disks vs Cylindrical shells: Deciding which to use

Region	Rotate about	Formula/Method
$y = f(x)$ $a \leq x \leq b$	x − axis	$\int_a^b \pi[f(x)]^2 \, dx$ disks
$y = f(x)$ $a \leq x \leq b$	y − axis	$\int_a^b 2\pi x f(x) \, dx$ shells
$x = f(y)$ $c \leq y \leq d$	y − axis	$\int_c^d \pi[f(y)]^2 \, dy$ disks
$x = f(y)$ $c \leq y \leq d$	x − axis	$\int_c^d 2\pi y f(y) \, dy$ shells

Note: In the situations considered above, we integrate with respect to the independent variable in the definition of the function.