5.5 Average Value of a Function

The average of a set of numbers $y_1, \dots, y_n$ is

$$A_n = \frac{y_1 + \cdots + y_n}{n}$$

The average value of a function $f(x)$ on $[a, b]$ cannot be computed by adding all the values of $f(x)$ because there are infinitely many of them. However, we can sample the values of $f(x)$ at equally spaced points $x_1, \dots, x_n$ in the interval and form the average of the numbers $y_1 = f(x_1), \dots, y_n = f(x_n)$.

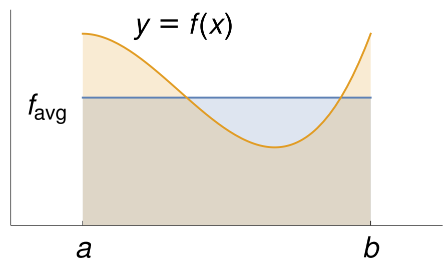
$$A_n = \frac{f(x_1) + \cdots + f(x_n)}{n} = \frac{1}{b-a}(f(x_1) + \cdots + f(x_n))\frac{b-a}{n} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = (b-a)/n$. The more points we use, the better A_n will represent the true "average" of $f(x)$. This leads us to define the average f_{avg} of $f(x)$ on $[a, b]$ to be the limit of A_n as $n \to \infty$. This limit can be calculated as an integral.

$$f_{\text{avg}} = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

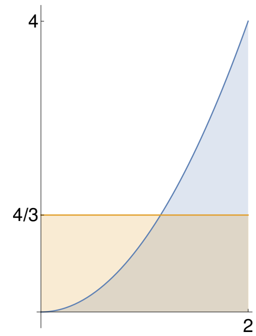
This formula gives us a universal way to interpret the integral of a function $f(x)$ on $[a, b]$: it is the average value of $f(x)$ times the length of the interval. In particular, if $f(x)$ is non-negative, then

$$\begin{aligned} \text{area under } f &= \int_a^b f(x)\,dx \\ &= (b-a)f_{\text{avg}} \\ &= \text{area of rectangle with height } f_{\text{avg}} \\ &\quad \text{and base } (b-a) \end{aligned}$$



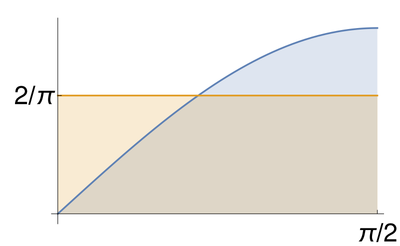
If the curved upper boundary of the region under the graph of f fluidly settles into a horizontal line, then the height of that line is f_{avg}.

Example: Find the average value of $f(x) = x^2$ on $[0, 2]$.

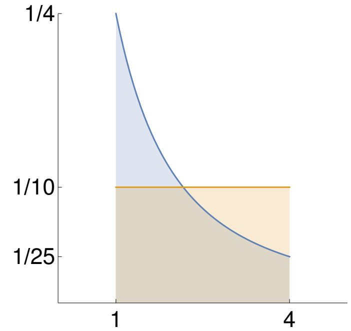


Solution: $f_{\text{avg}} = \dfrac{1}{2-0}\displaystyle\int_0^2 x^2\,dx = \dfrac{1}{2}\cdot\dfrac{1}{3}x^3\Big|_0^2 = \dfrac{4}{3}$

Example: Find the average value of $\sin(x)$ on $[0, \pi/2]$.



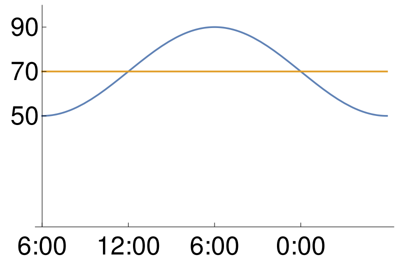
Solution: $A = \dfrac{1}{\pi/2}\displaystyle\int_0^{\pi/2} \sin(x)\,dx = -\dfrac{2}{\pi}\cos(x)\Big|_0^{\pi/2} = \dfrac{2}{\pi} = .63662.$



Example: Find the average value of $f(x) = \dfrac{1}{(x+1)^2}$ on $[1, 4]$.

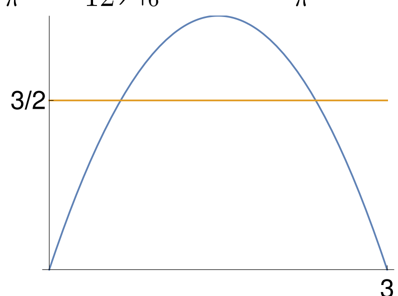
Solution: $f_{\text{avg}} = \dfrac{1}{4-1}\displaystyle\int_1^4 (x+1)^{-2}\,dx = -\dfrac{1}{3}(x+1)^{-1}\Big|_1^4 = -\dfrac{1}{15} + \dfrac{1}{6} = \dfrac{-2+5}{30} = \dfrac{1}{10}.$

Example: The temperature during the summer is modeled by the function $T(t) = 70 - 20\cos\dfrac{\pi t}{12}$, where t is hours after 6:00 am. Find the average temperature from noon until midnight.



Solution: $T_{\text{avg}} = \dfrac{1}{12}\displaystyle\int_6^{18} 70 - 20\cos\dfrac{\pi t}{12}\,dt = \dfrac{1}{12}\left(70t - 20\cdot\dfrac{12}{\pi}\sin\dfrac{\pi t}{12}\right)\Big|_6^{18} = 70 + \dfrac{40}{\pi} = 82.73.$

Example: The density of a rod is given by $\rho(x) = x(3-x)$ for $0 \le x \le 3$. Find the average density of the rod.



Solution: $\rho_{\text{avg}} = \dfrac{1}{3-0}\displaystyle\int_0^3 3x - x^2\,dx = \dfrac{1}{3}\left(\dfrac{3}{2}x^2 - \dfrac{1}{3}x^3\right)\Big|_0^3 = \dfrac{9}{2} - 3 = \dfrac{3}{2}.$

Mean Value Theorem for Integrals. If $f(x)$ is continuous on $[a, b]$, then there is a number c in $[a, b]$ such that $f(c) = f_{\text{avg}}$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

If we could only find this number c, we would have an easy way to compute the integral, $\int_a^b f(x)\,dx = f(c)(b-a)$!

Example: Find c for $f(x) = x^2$ on $[0, 2]$

Solution: $f_{\text{avg}} = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$. The value of c that satisfies $f(c) = c^2 = \frac{4}{3}$ is $c = \frac{2}{\sqrt{3}}$.

Example: Prove the Mean Value Theorem for Integrals from the Mean Value Theorem for Derivatives.

Solution: Let $F(x) = \int_a^x f(t)\,dt$. By the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for $a \leq x \leq b$. The Mean Value Theorem for Derivatives states there is a number c in $[a, b]$ such that

$$F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a}\int_a^b f(x), dx$$

The left side of this equation is $f(c)$ and the right side is f_{avg}.