Lecture 3 : Limit of a Function

Click on this symbol to view an interactive demonstration in Wolfram Alpha.

Limit of a Function

Consider the behavior of the values of $f(x) = x^2$ as x gets closer and closer ... and closer .... to 3.

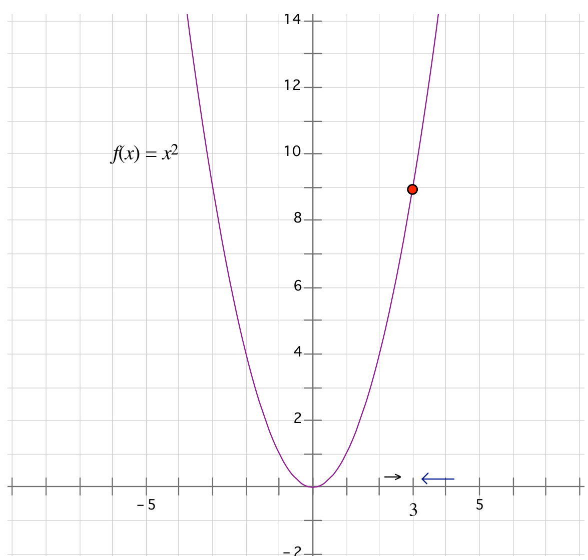
Example Let $f(x) = x^2$. The table below shows the behavior of the values of $f(x)$ as x approaches 3 from the left and from the right.

x	$f(x) = x^2$	x	$f(x) = x^2$
2	4	4.0	16
2.5	6.25	3.5	12.25
2.9	8.41	3.1	9.61
2.95	8.70	3.05	9.3
2.99	8.94	3.01	9.06
2.995	8.97	3.005	9.03
2.999	8.99	3.001	9.006

We see that the values of $f(x) = x^2$ get closer to __________ as the sequence of values of x approaches 3. We also say that as x tends to 3, $f(x) = x^2$ tends to ________ or we abbreviate the statement with the notation:

$$f(x) \to \_\_\_\_\_\_\_ \quad \text{as} \quad x \to 3.$$

We can also use the graph below to see the behavior of the values of $f(x)$ as x approaches 3:



Definition We write

$$\lim_{x \to a} f(x) = L$$

and say "The limit of $f(x)$, as x approaches a, equals L", if we can make the value of $f(x)$ as close as we like to L, by taking x sufficiently close to a(on either side) but not equal to a.

Note A Table of values like the one shown above for $f(x) = x^2$ is useful for predicting what the limit might be, but may give the wrong impression. (See the example where $f(x) = \sin(1/x)$ at the end of this set of notes). <u>For now an accurate graph is the most reliable method we have to find limits.</u> In the next sections we will use a catalogue of well known limits together with some rules to calculate limits of more complicated functions. We give an outline of an algebraic proof that that $\lim_{x\to 3} x^2 = 9$ at the end of this set of lecture notes.

Example Use the graph of $y = x^2$ above to evaluate the following limits:

$\lim_{x\to 3} x^2 =$ $\qquad$ $\lim_{x\to 2} x^2 =$ $\qquad$.

- Roughly speaking, the statement $\lim_{x\to a} f(x) = L$ means that as the values of x get close to (but not equal to) a, the values of $f(x)$ get closer and closer to L.
- The value of the function $f(x)$ at the point $x = a$, plays no role in determining the value of the limit of the function at $x = a$ (if it exists), since we only take into account the behavior of a function near the point $x = a$ to determine if it has a limit of not. (see the example below).

Example Let

$$g(x) = \begin{cases} x^2 & x \neq 3 \\ 0 & x = 3 \end{cases}$$

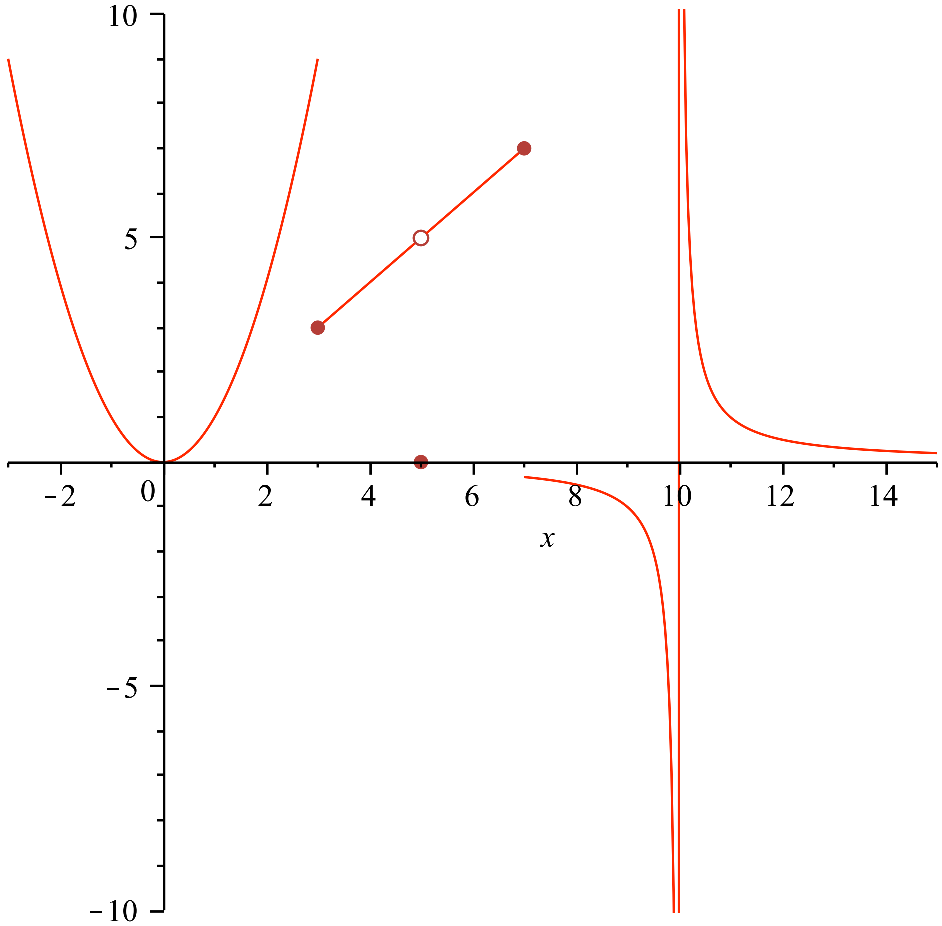
(a) Draw the graph of this function and use the graph to find

$$\lim_{x\to 3} g(x)$$

- Note that the value of $\lim_{x\to 3} g(x) \neq g(3)$ above.
- If the values of two functions, $f(x)$ and $g(x)$ are the same except at $x = a$, then they have the same limit as x approaches a if that limit exists, i.e. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if it exists. (for example $f(x)$ and $g(x)$ above.)
- Sometimes the values of a function do not have a limit as x approaches a number a and, in this case, we say $\lim_{x\to a} f(x)$ does not exist. We will examine a number of ways in which this can happen below. (see the function $k(x)$ shown below at $x = 3, 7, 10$.)
- The value of the function $f(x)$ at the point $x = a$, plays no role in whether the limit exists or not, since we only take into account the behavior of a function near the point $x = a$ to determine if it has a limit of not (Sometimes $\lim_{x\to a} f(x)$ exists for values of a which are not in the domain of f [e.g. $g_1(x) = \frac{(x-3)x^2}{(x-3)} = \begin{cases} x^2 & x \neq 3 \\ \text{undefined} & x = 3 \end{cases}$. Also check out $f(x) = x^2 \sin(1/x)$ next lecture.])

Example Consider the graph shown below of the function

$$k(x) = \begin{cases} x^2 & -3 < x < 3 \\ x & 3 \le x < 5 \\ 0 & x = 5 \\ x & 5 < x \le 7 \\ \frac{1}{x-10} & x > 7 \end{cases}$$



The limit, $\lim_{x \to 0} k(x)$, when it exists will be the (unique) y-value that you approach as you travel along the graph of the function, from both sides.

(a) What is $\lim_{x \to 0} k(x)$?

(b) What happens at $x = 3$. Is there a unique number L so that we can make the value of $f(x)$ as close as we like to L, by taking x sufficiently close to $a = 3$(on either side) but not equal to $a = 3$? In other words, does $\lim_{x \to 3} k(x)$ exist?

Left and Right Hand Limits

Definition We write $\lim_{x\to a^-} f(x) = L$ and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a with x less than a. We say $\lim_{x\to a^+} f(x) = L$ and say the right-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a with x greater than a.

Note : $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

(c) Evaluate

$$\lim_{x\to 5} k(x)$$

.

(d) What is $\lim_{x\to 7^-} k(x)$?

What is $\lim_{x\to 7^+} k(x)$?

Does $\lim_{x\to 7} k(x)$ exist?

(e) Does $\lim_{x\to 10} k(x)$ exist?

Infinite Limits

Definition: We write $\lim_{x\to a^-} f(x) = -\infty$ and say the left-hand limit of $f(x)$ as x approaches a is equal to $-\infty$ if we can make the values of $f(x)$ arbitrarily large and negative by taking x sufficiently close to a with x less than a. Similar definitions are used for the one sided infinite limits:

$$\lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

Definition The line $x = a$ is a vertical asymptote to the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^-} f(x) = \infty, \quad \lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^+} f(x) = -\infty.$$

Definition If $\lim_{x\to a^-} f(x) = \infty = \lim_{x\to a^+} f(x)$, then we say

$$\lim_{x\to a} f(x) = \infty.$$

Sinilarly if $\lim_{x\to a^-} f(x) = -\infty = \lim_{x\to a^+} f(x)$, then we say

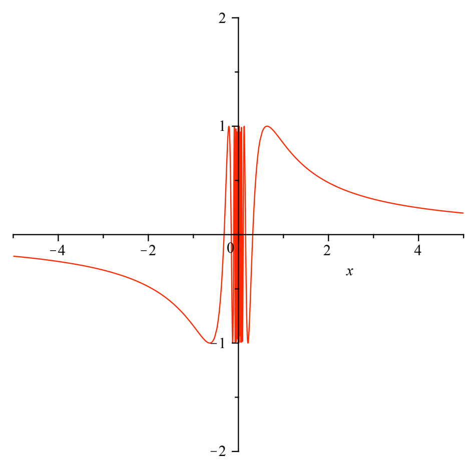
$$\lim_{x\to a} f(x) = -\infty.$$

(f) Does the graph of $k(x)$ above have a vertical asymptote? If so what is the equation of the vertical asymptote?

(g) Determine the infinite limits $\lim_{x\to 10^+} k(x)$ and $\lim_{x\to 10^-} k(x)$. (Say whether the limit is ∞ or $-\infty$.)

In the following example, we will see why a table of function values may be misleading when calculating limits.

Example The graph of $f(x) = \sin(1/x)$ is shown below. If we look at the behavior of the curve as x approaches 0, we see that the graph oscillates between -1 and +1 with increasing frequency. Since the $y-$values on the graph do not approach a unique y-value L as x approaches 0, we have that $\lim_{x\to 0}\sin(1/x)$ does not exist.



In this case, we can find two infinite sequences of x values both approaching 0, but giving different impressions of what happens the function values as x approaches 0.
Fill in the values of $\sin(1/x)$ for both sequences of x-values approaching 0 below.

x	$f(x) = \sin(1/x)$	x	$f(x) = \sin(1/x)$
$2/3\pi$		$2/\pi$	
$2/7\pi$		$2/5\pi$	
$2/11\pi$		$2/9\pi$	
$2/15\pi$		$2/13\pi$	
$2/(4n-1)\pi$		$2/(4n+1)\pi$	

Note For any function $f(x)$, if $\lim_{x\to a} f(x)$ exists, then we cannot find two infinite sequences of x-values approaching 0 for which the corresponding function values approach different numbers

Appendix

How do we prove algebraically that we can make the values of x^2 as close as we like to 9, by taking x sufficiently close to 3(on either side) but not equal to 3.

The following statement guarantees it:

Given any number of decimal places, say n of them, I can always say that if x is equal to 3 up to $n+1$ decimal places, then x^2 is equal to 9 up to n decimal places. For example if $x = 3+h$, where $h < .00001$, then $x^2 = 9 + 2h + h^2$ and $2h + h^2 < .0001$, hence x^2 is certainly equal to 9 up to 3 decimal places.

So if I take a sequence of x values approaching 3, as the values of x get closer and closer to 3, the values of $f(x) = x^2$ are guaranteed to be equal to 9 up to 10 decimal places, 100 decimal places , 1000 decimal places, as the values of x get within 11, 101 and 102 decimal places of 3 respectively. Hence, for every sequence of values of x approaching 3, the values of $f(x) = x^2$ approach 9.