Lecture 3 : Limit of a Function

Click on this symbol to view an interactive demonstration in Wolfram Alpha.

Limit of a Function

Consider the behavior of the values of $f(x) = x^2$ as $x$ gets closer and closer ... and closer .... to 3.

Example Let $f(x) = x^2$. The table below shows the behavior of the values of $f(x)$ as $x$ approaches 3 from the left and from the right.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$x$</th>
<th>$f(x) = x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>4.0</td>
<td>16</td>
</tr>
<tr>
<td>2.5</td>
<td>6.25</td>
<td>3.5</td>
<td>12.25</td>
</tr>
<tr>
<td>2.9</td>
<td>8.41</td>
<td>3.1</td>
<td>9.61</td>
</tr>
<tr>
<td>2.95</td>
<td>8.70</td>
<td>3.05</td>
<td>9.3</td>
</tr>
<tr>
<td>2.99</td>
<td>8.94</td>
<td>3.01</td>
<td>9.06</td>
</tr>
<tr>
<td>2.995</td>
<td>8.97</td>
<td>3.005</td>
<td>9.03</td>
</tr>
<tr>
<td>2.999</td>
<td>8.99</td>
<td>3.001</td>
<td>9.006</td>
</tr>
</tbody>
</table>

We see that the values of $f(x) = x^2$ get closer to 16 as the sequence of values of $x$ approaches 3. We also say that as $x$ tends to 3, $f(x) = x^2$ tends to 16 or we abbreviate the statement with the notation:

$$f(x) \to 16 \text{ as } x \to 3.$$ 

We can also use the graph below to see the behavior of the values of $f(x)$ as $x$ approaches 3:

Definition We write

$$\lim_{x \to \alpha} f(x) = L$$

and say “The limit of $f(x)$, as $x$ approaches $\alpha$, equals $L$”, if we can make the value of $f(x)$ as close as we like to $L$, by taking $x$ sufficiently close to $\alpha$ (on either side) but not equal to $\alpha$.

Note A Table of values like the one shown above for $f(x) = x^2$ is useful for predicting what the limit might be, but may give the wrong impression. (See the example where $f(x) = \sin(1/x)$ at the end of this set of notes). For now an accurate graph is the most reliable method we have to find limits. In
the next sections we will use a catalogue of well known limits together with some rules to calculate limits of more complicated functions. We give an outline of an algebraic proof that that $\lim_{x \to 3} x^2 = 9$ at the end of this set of lecture notes.

**Example** Use the graph of $y = x^2$ above to evaluate the following limits:

\[
\lim_{x \to 3} x^2 = \quad \lim_{x \to -2} x^2 = \quad .
\]

- Roughly speaking, the statement $\lim_{x \to a} f(x) = L$ means that as the values of $x$ get close to (but not equal to) $a$, the values of $f(x)$ get closer and closer to $L$.
- The value of the function $f(x)$ at the point $x = a$, plays no role in determining the value of the limit of the function at $x = a$ (if it exists), since we only take into account the behavior of a function near the point $x = a$ to determine if it has a limit of not. (see the example below).

**Example** Let

\[
g(x) = \begin{cases} 
  x^2 & x \neq 3 \\
  0 & x = 3 
\end{cases}
\]

(a) Draw the graph of this function and use the graph to find 

\[
\lim_{x \to 3} g(x)
\]

- Note that the value of $\lim_{x \to 3} g(x) \neq g(3)$ above.
- If the values of two functions, $f(x)$ and $g(x)$ are the same except at $x = a$, then they have the same limit as $x$ approaches $a$ if that limit exists, i.e. $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ if it exists. (for example $f(x)$ and $g(x)$ above.)
- Sometimes the values of a function do not have a limit as $x$ approaches a number $a$ and, in this case, we say $\lim_{x \to a} f(x)$ does not exist. We will examine a number of ways in which this can happen below. (see the function $k(x)$ shown below at $x = 3, 7, 10$.)
- The value of the function $f(x)$ at the point $x = a$, plays no role in whether the limit exists or not, since we only take into account the behavior of a function near the point $x = a$ to determine if it has a limit of not (Sometimes $\lim_{x \to a} f(x)$ exists for values of $a$ which are not in the domain of $f$ [e.g. $g_1(x) = \frac{(x-3)x^2}{(x-3)} = \begin{cases} 
  x^2 & x \neq 3 \\
  \text{undefined} & x = 3 
\end{cases}$. Also check out $f(x) = x^2 \sin(1/x)$ next lecture.])
Example  Consider the graph shown below of the function

\[ k(x) = \begin{cases} 
  x^2 & -3 < x < 3 \\
  x & 3 \leq x < 5 \\
  0 & x = 5 \\
  x & 5 < x \leq 7 \\
  \frac{1}{x-10} & x > 7 
\end{cases} \]

The limit, \( \lim_{x \to 0} k(x) \), when it exists will be the (unique) y-value that you approach as you travel along the graph of the function, from both sides.

(a) What is \( \lim_{x \to 0} k(x) \)?

(b) What happens at \( x = 3 \). Is there a unique number \( L \) so that we can make the value of \( f(x) \) as close as we like to \( L \), by taking \( x \) sufficiently close to \( a = 3 \)(on either side) but not equal to \( a = 3 \)? In other words, does \( \lim_{x \to 3} k(x) \) exist?
Left and Right Hand Limits

**Definition**  We write \( \lim_{x \to a^-} f(x) = L \) and say the left-hand limit of \( f(x) \) as \( x \) approaches \( a \) is equal to \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) with \( x \) less than \( a \). We say \( \lim_{x \to a^+} f(x) = L \) and say the right-hand limit of \( f(x) \) as \( x \) approaches \( a \) is equal to \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) with \( x \) greater than \( a \).

**Note:** \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^-} f(x) = L \) and \( \lim_{x \to a^+} f(x) = L \).

(c) Evaluate \( \lim_{x \to 5} k(x) \).

(d) What is \( \lim_{x \to 7^-} k(x) \)?

What is \( \lim_{x \to 7^+} k(x) \)?

Does \( \lim_{x \to 7} k(x) \) exist?

(e) Does \( \lim_{x \to 10} k(x) \) exist?

### Infinite Limits

**Definition:** We write \( \lim_{x \to a^-} f(x) = -\infty \) and say the left-hand limit of \( f(x) \) as \( x \) approaches \( a \) is equal to \( -\infty \) if we can make the values of \( f(x) \) arbitrarily large and negative by taking \( x \) sufficiently close to \( a \) with \( x \) less than \( a \). Similar definitions are used for the one sided infinite limits:

\[
\lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = \infty \quad \lim_{x \to a^+} f(x) = -\infty.
\]

**Definition** The line \( x = a \) is a vertical asymptote to the curve \( y = f(x) \) if at least one of the following is true:

\[
\lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^+} f(x) = \infty \quad \lim_{x \to a^+} f(x) = -\infty.
\]

**Definition** If \( \lim_{x \to a^-} f(x) = \infty = \lim_{x \to a^+} f(x) \), then we say

\[
\lim_{x \to a} f(x) = \infty.
\]
Similarly if \( \lim_{x \to a^-} f(x) = -\infty = \lim_{x \to a^+} f(x) \), then we say

\[
\lim_{x \to a} f(x) = -\infty.
\]

(f) Does the graph of \( k(x) \) above have a vertical asymptote? If so what is the equation of the vertical asymptote?

(g) Determine the infinite limits \( \lim_{x \to 10^+} k(x) \) and \( \lim_{x \to 10^-} k(x) \). (Say whether the limit is \( \infty \) or \(-\infty\).)

In the following example, we will see why a table of function values may be misleading when calculating limits.

**Example** The graph of \( f(x) = \sin(1/x) \) is shown below. If we look at the behavior of the curve as \( x \) approaches 0, we see that the graph oscillates between -1 and +1 with increasing frequency. Since the \( y \)-values on the graph do not approach a unique \( y \)-value \( L \) as \( x \) approaches 0, we have that \( \lim_{x \to 0} \sin(1/x) \) does not exist.

In this case, we can find two infinite sequences of \( x \) values both approaching 0, but giving different impressions of what happens the function values as \( x \) approaches 0. Fill in the values of \( \sin(1/x) \) for both sequences of \( x \)-values approaching 0 below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sin(1/x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2/3\pi )</td>
<td>( 2/\pi )</td>
</tr>
<tr>
<td>( 2/7\pi )</td>
<td>( 2/5\pi )</td>
</tr>
<tr>
<td>( 2/11\pi )</td>
<td>( 2/9\pi )</td>
</tr>
<tr>
<td>( 2/15\pi )</td>
<td>( 2/13\pi )</td>
</tr>
<tr>
<td>( 2/(4n - 1)\pi )</td>
<td>( 2/(4n + 1)\pi )</td>
</tr>
</tbody>
</table>

**Note** For any function \( f(x) \), if \( \lim_{x \to a} f(x) \) exists, then we cannot find two infinite sequences of \( x \)-values approaching 0 for which the corresponding function values approach different numbers.
Appendix

How do we prove algebraically that we can make the values of \( x^2 \) as close as we like to 9, by taking \( x \) sufficiently close to 3(on either side) but not equal to 3.

The following statement guarantees it:

Given any number of decimal places, say \( n \) of them, I can always say that if \( x \) is equal to 3 up to \( n+1 \) decimal places, then \( x^2 \) is equal to 9 up to \( n \) decimal places. For example if \( x = 3 + h \), where \( h < .00001 \), then \( x^2 = 9 + 2h + h^2 \) and \( 2h + h^2 < .0001 \), hence \( x^2 \) is certainly equal to 9 up to 3 decimal places.

So if I take a sequence of \( x \) values approaching 3, as the values of \( x \) get closer and closer to 3, the values of \( f(x) = x^2 \) are guaranteed to be equal to 9 up to 10 decimal places, 100 decimal places, 1000 decimal places, as the values of \( x \) get within 11, 101 and 102 decimal places of 3 respectively. Hence, for every sequence of values of \( x \) approaching 3, the values of \( f(x) = x^2 \) approach 9.