Lecture 4 : Calculating Limits using Limit Laws

Click on this symbol ⤚ to view an interactive demonstration in Wolfram Alpha.

Using the definition of the limit, \( \lim_{x \to a} f(x) \), we can derive many general laws of limits, that help us to calculate limits quickly and easily. The following rules apply to any functions \( f(x) \) and \( g(x) \) and also apply to left and right sided limits:

Suppose that \( c \) is a constant and the limits \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist (meaning they are finite numbers). Then

1. \( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \); (the limit of a sum is the sum of the limits).

2. \( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \); (the limit of a difference is the difference of the limits).

3. \( \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) \); (the limit of a constant times a function is the constant times the limit of the function).

4. \( \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \); (The limit of a product is the product of the limits).

5. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) if \( \lim_{x \to a} g(x) \neq 0 \); (the limit of a quotient is the quotient of the limits provided that the limit of the denominator is not 0)

**Example** If I am given that
\[
\lim_{x \to 2} f(x) = 2, \quad \lim_{x \to 2} g(x) = 5, \quad \lim_{x \to 2} h(x) = 0.
\]
find the limits that exist (are a finite number):

(a) \( \lim_{x \to 2} \frac{2f(x) + h(x)}{g(x)} = \frac{\lim_{x \to 2} (2f(x) + h(x))}{\lim_{x \to 2} g(x)} \) since \( \lim_{x \to 2} g(x) \neq 0 \)
\[
= \frac{2 \lim_{x \to 2} f(x) + \lim_{x \to 2} h(x)}{\lim_{x \to 2} g(x)} = \frac{2(2) + 0}{5} = \frac{4}{5}
\]

(b) \( \lim_{x \to 2} \frac{f(x)}{h(x)} \)

(c) \( \lim_{x \to 2} \frac{f(x)h(x)}{g(x)} \)

**Note 1** If \( \lim_{x \to a} g(x) = 0 \) and \( \lim_{x \to a} f(x) = b \), where \( b \) is a finite number with \( b \neq 0 \), Then: the values of the quotient \( \frac{f(x)}{g(x)} \) can be made arbitrarily large in absolute value as \( x \to a \) and thus
the limit does not exist.
If the values of \( \frac{f(x)}{g(x)} \) are positive as \( x \to a \) in the above situation, then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \infty \),
If the values of \( \frac{f(x)}{g(x)} \) are negative as \( x \to a \) in the above situation, then \( \lim_{x \to a} \frac{f(x)}{g(x)} = -\infty \),
If on the other hand, if \( \lim_{x \to a} g(x) = 0 = \lim_{x \to a} f(x) \), we cannot make any conclusions about the limit.

**Example** Find \( \lim_{x \to \pi} \frac{\cos x}{x - \pi} \).

As \( x \) approaches \( \pi \) from the left, \( \cos x \) approaches a finite number \(-1\).
As \( x \) approaches \( \pi \) from the left, \( x - \pi \) approaches 0.
Therefore as \( x \) approaches \( \pi \) from the left, the quotient \( \frac{\cos x}{x - \pi} \) approaches \( \infty \) in absolute value.
The values of both \( \cos x \) and \( x - \pi \) are negative as \( x \) approaches \( \pi \) from the left, therefore
\[
\lim_{x \to \pi} \frac{\cos x}{x - \pi} = \infty.
\]

More powerful laws of limits can be derived using the above laws 1-5 and our knowledge of some basic functions. The following can be proven reasonably easily (we are still assuming that \( c \) is a constant and \( \lim_{x \to a} f(x) \) exists);

6. \( \lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \), where \( n \) is a positive integer (we see this using rule 4 repeatedly).
7. \( \lim_{x \to a} c = c \), where \( c \) is a constant (easy to prove from definition of limit and easy to see from the graph, \( y = c \)).
8. \( \lim_{x \to a} x = a \), (follows easily from the definition of limit)
9. \( \lim_{x \to a} x^n = a^n \) where \( n \) is a positive integer (this follows from rules 6 and 8).
10. \( \lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \), where \( n \) is a positive integer and \( a > 0 \) if \( n \) is even. (proof needs a little extra work and the binomial theorem)
11. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \) assuming that the \( \lim_{x \to a} f(x) > 0 \) if \( n \) is even. (We will look at this in more detail when we get to continuity)

**Example** Evaluate the following limits and justify each step:

(a) \( \lim_{x \to 3} \frac{x^3+2x^2-x+1}{x-1} \)

(b) \( \lim_{x \to 1} \sqrt{x+1} \)
(c) Determine the infinite limit (see note 1 above, say if the limit is $\infty$, $-\infty$ or D.N.E.) 
$$\lim_{x \to -2} \frac{x+1}{x-2}.$$ 

Polynomial and Rational Functions

Please review the relevant parts of Lectures 3, 4 and 7 from the Algebra/Precalculus review page. This demonstration will help you visualize some rational functions:

Direct Substitution (Evaluation) Property  If $f$ is a polynomial or a rational function and $a$ is in the domain of $f$, then $\lim_{x \to a} f(x) = f(a)$. This follows easily from the rules shown above. (Note that this is the case in part (a) of the example above)

if $f(x) = \frac{P(x)}{Q(x)}$ is a rational function where $P(x)$ and $Q(x)$ are polynomials with $Q(a) = 0$, then:

- If $P(a) \neq 0$, we see from note 1 above that $\lim_{x \to a} \frac{P(x)}{Q(x)} = \pm \infty$ or D.N.E. and is not equal to $\pm \infty$.
- If $P(a) = 0$ we can cancel a factor of the polynomial $P(x)$ with a factor of the polynomial $Q(x)$ and the resulting rational function may have a finite limit or an infinite limit or no limit at $x = a$. The limit of the new quotient as $x \to a$ is equal to $\lim_{x \to a} \frac{P(x)}{Q(x)}$ by the following observation which we made in the last lecture:

Note 2: If $h(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} h(x) = \lim_{x \to a} g(x)$ provided the limits exist.

Example  Determine if the following limits are finite, equal to $\pm \infty$ or D.N.E. and are not equal to $\pm \infty$:

(a) $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

(b) $\lim_{x \to 1} \frac{x^2-x-6}{x-1}$.

(c) Which of the following is true:
1. $\lim_{x \to -1} \frac{x^2-x-6}{x-1} = +\infty$,  
2. $\lim_{x \to -1} \frac{x^2-x-6}{x-1} = -\infty$,  
3. $\lim_{x \to -1} \frac{x^2-x-6}{x-1}$ D.N.E. and is not $\pm \infty$, 

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Example Evaluate the limit (finish the calculation)

\[
\lim_{h \to 0} \frac{(3 + h)^2 - (3)^2}{h}.
\]

\[
\lim_{h \to 0} \frac{(3 + h)^2 - (3)^2}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} 6 + h = 6.
\]

Example Evaluate the following limit:

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 25} - 5}{x^2}.
\]

Recall also our observation from the last day which can be proven rigorously from the definition (this is good to keep in mind when dealing with piecewise defined functions):

Theorm \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \).

Example Evaluate the limit if it exists:

\[
\lim_{x \to -2} \frac{3x + 6}{|x + 2|}.
\]

The following theorems help us calculate some important limits by comparing the behavior of a function with that of other functions for which we can calculate limits:
Theorem  If \( f(x) \leq g(x) \) when \( x \) is near \( a \) (except possible at \( a \)) and the limits of \( f(x) \) and \( g(x) \) both exist as \( x \) approaches \( a \), then
\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).
\]

The Sandwich (squeeze) Theorem  If \( f(x) \leq g(x) \leq h(x) \) when \( x \) is near \( a \) (except possibly at \( a \)) and \[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L
\]
then
\[
\lim_{x \to a} g(x) = L.
\]

Recall last day, we saw that \( \lim_{x \to 0} \sin(1/x) \) does not exist because of how the function oscillates near \( x = 0 \). However we can see from the graph below and the above theorem that \( \lim_{x \to 0} x^2 \sin(1/x) = 0 \), since the graph of the function is sandwiched between \( y = -x^2 \) and \( y = x^2 \):

\[\text{Graph of} \quad y = x^2 \sin(1/x) \]

Example  Calculate the limit \( \lim_{x \to 0} x^2 \sin \frac{1}{x} \).
We have \(-1 \leq \sin(1/x) \leq 1\) for all \( x \), multiplying across by \( x^2 \) (which is positive), we get \(-x^2 \leq x^2 \sin(1/x) \leq x^2\) for all \( x \),
Using the Sandwich theorem, we get
\[
0 = \lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \sin(1/x) \leq \lim_{x \to 0} x^2 = 0
\]
Hence we can conclude that
\[
\lim_{x \to 0} x^2 \sin(1/x) = 0.
\]

Example  Decide if the following limit exists and if so find its values:
\[
\lim_{x \to 0} x^{100} \cos^2(\pi/x)
\]
Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

1. \( \lim_{x \to 1} x^4 + 2x^3 + x^2 + 3 \)

2. \( \lim_{x \to 2} \frac{x^2 - 2x + 2}{(x-2)^2} \).

3. \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{|x|} \right) \).

4. \( \lim_{x \to 0} \frac{|x|}{x^2 + x + 10} \).

5. \( \lim_{h \to 0} \frac{\sqrt{1+h} - 2}{h} \).

6. If \( 2x \leq g(x) \leq x^2 - x + 2 \) for all \( x \), evaluate \( \lim_{x \to 1} g(x) \).

7. Determine if the following limit is finite, \( \pm \infty \) or D.N.E. and is not \( \pm \infty \).

\[
\lim_{x \to 1^-} \frac{(x - 3)(x + 2)}{(x - 1)(x - 2)}.
\]
Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

(1) \( \lim_{x \to 1} x^4 + 2x^3 + x^2 + 3 \)

Since this is a polynomial function, we can calculate the limit by direct substitution:

\[
\lim_{x \to 1} x^4 + 2x^3 + x^2 + 3 = 1^4 + 2(1)^3 + 1^2 + 3 = 7.
\]

(2) \( \lim_{x \to 2} \frac{x^2 - 3x + 2}{(x-2)^2} \)

This is a rational function, where both numerator and denominator approach 0 as \( x \) approaches 2. We factor the numerator to get

\[
\lim_{x \to 2} \frac{x^2 - 3x + 2}{(x-2)^2} = \lim_{x \to 2} \frac{(x-1)(x-2)}{(x-2)^2}
\]

After cancellation, we get

\[
\lim_{x \to 2} \frac{(x-1)(x-2)}{(x-2)^2} = \lim_{x \to 2} \frac{(x-1)}{(x-2)}.
\]

Now this is a rational function where the numerator approaches 1 as \( x \to 2 \) and the denominator approaches 0 as \( x \to 2 \). Therefore

\[
\lim_{x \to 2} \frac{(x-1)}{(x-2)}
\]

does not exist.

We can analyze this limit a little further, by checking out the left and right hand limits at 2. As \( x \) approaches 2 from the left, the values of \( (x-1) \) are positive (approaching a constant 1) and the values of \( (x-2) \) are negative (approaching 0). Therefore the values of \( \frac{(x-1)}{(x-2)} \) are negative and become very large in absolute value. Therefore

\[
\lim_{x \to 2^-} \frac{(x-1)}{(x-2)} = -\infty.
\]

Similarly, you can show that

\[
\lim_{x \to 2^-} \frac{(x-1)}{(x-2)} = +\infty,
\]

and therefore the graph of \( y = \frac{(x-1)}{(x-2)} \) has a vertical asymptote at \( x = 2 \).

(check it out on your calculator)

(3) \( \lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{|x|} \right) \)

Let \( f(x) = \frac{1}{x} - \frac{1}{|x|} \). We write this function as a piecewise defined function:

\[
f(x) = \begin{cases} 
\frac{1}{x} - \frac{1}{x} = 0 & x > 0 \\
\frac{1}{x} + \frac{1}{x} = \frac{2}{x} & x \leq 0 
\end{cases}
\]
lim_{x \to 0} \left( \frac{\frac{1}{x} - \frac{1}{|x|}}{x^2 + x + 10} \right) exists only if the left and right hand limits exist and are equal. 
lim_{x \to 0^+} \left( \frac{\frac{1}{x} - \frac{1}{|x|}}{x^2 + x + 10} \right) = \lim_{x \to 0^-} \left( \frac{\frac{1}{x} - \frac{1}{|x|}}{x^2 + x + 10} \right) = \lim_{x \to 0^-} \frac{\frac{2}{x}}{2} = -\infty.
Since the limits do not match, we have 
\lim_{x \to 0} \left( \frac{\frac{1}{x} - \frac{1}{|x|}}{x^2 + x + 10} \right) \text{ D.N.E.}

(4) \quad \lim_{x \to 0} \frac{|x|}{x^2 + x + 10}.

Since \lim_{x \to 0} x^2 + x + 10 = 10 \neq 0, we have 
\lim_{x \to 0} \frac{|x|}{x^2 + x + 10} = \frac{\lim_{x \to 0} |x|}{\lim_{x \to 0} (x^2 + x + 10)} = \lim_{x \to 0} \frac{|x|}{10}.
Now 
|x| = \begin{cases} 
x & x > 0 
-x & x \leq 0
\end{cases}.
Clearly \lim_{x \to 0^+} |x| = 0 = \lim_{x \to 0^-} |x|. Hence \lim_{x \to 0} |x| = 0 and 
\lim_{x \to 0} \frac{|x|}{x^2 + x + 10} = \lim_{x \to 0} \frac{|x|}{10} = \frac{0}{10} = 0.

(5) \quad \lim_{h \to 0} \sqrt{\frac{4 + h}{4}} - \frac{2}{h}.
Since \lim_{h \to 0} \sqrt{\frac{4 + h}{4}} - \frac{2}{h} = 0 \neq \lim_{h \to 0} h, we cannot determine whether this limit exists or not from the limit laws without some transformation. We have 
\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h} = \lim_{h \to 0} \frac{(\sqrt{4 + h} - 2)(\sqrt{4 + h} + 2)}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{(\sqrt{4 + h})^2 - 4}{h(\sqrt{4 + h} + 2)}
= \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{h}{h(\sqrt{4 + h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{4}.

(6) \quad \text{If } 2x \leq g(x) \leq x^2 - x + 2 \text{ for all } x, \text{ evaluate } \lim_{x \to -1} g(x).
We use the Sandwich theorem here. Since \lim_{x \to -1} 2x \leq \lim_{x \to -1} g(x) \leq \lim_{x \to -1} (x^2 - x + 2),
therefore 
\lim_{x \to -1} 2x \leq \lim_{x \to -1} g(x) \leq \lim_{x \to -1} (x^2 - x + 2),
and hence 
\lim_{x \to -1} g(x) = 2.
(7) Determine if the following limit is finite, $\pm \infty$ or D.N.E. and is not $\pm \infty$.

\[
\lim_{x \to 1^-} \frac{(x - 3)(x + 2)}{(x - 1)(x - 2)}.
\]

Let $P(x) = (x - 3)(x + 2)$ and $Q(x) = (x - 1)(x - 2)$. We have $P(1) = -6 \neq 0$ and $Q(1) = 0$. Therefore the values of $\frac{P(x)}{Q(x)} = \frac{(x-3)(x+2)}{(x-1)(x-2)}$ get larger in absolute value as $x$ approaches 1.

As $x$ approaches 1 from the left, $(x - 3) < 0$, $(x - 2) < 0$, $(x - 1) < 0$, and $(x + 2) > 0$, therefore the quotient $\frac{(x-3)(x+2)}{(x-1)(x-2)} < 0$ as $x$ approaches 1 from the left and therefore

\[
\lim_{x \to 1^-} \frac{(x - 3)(x + 2)}{(x - 1)(x - 2)} = -\infty.
\]