Derivative as a function

In the previous section we defined the derivative of a function f at a number a (when the function f is defined in an open interval containing a) to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

when this limit exists. This gives the **slope of the tangent** to the curve $y = f(x)$ when $x = a$

Example Last day we saw that if $f(x) = x^2 + 5x$, then $f'(a) = 2a + 5$ for any value of a. Therefore $f'(1) = 7, f'(2) = 9, f'(2.5) = 10$ etc....

The value of $f'(a)$ varies as the number a varies, hence f' is a function of a. We can change the variable from a to x to get a new function, called **The derivative of** f

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Domain of $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

($f'(x)$ is defined when f is defined in an open interval containing x and the above limit exists). Note that when calculating this limit for a particular value of x, $h \to 0$ and the value of x remains constant.

Note also that **if** x **is in the domain of** f', it must satisfy the following 3 conditions:

1. x must be in the domain of f.
2. $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ must exist at x.
3. f must be defined in an open interval containing x.

The domain of the function f' may be smaller than the domain of the function f since 2 or 3 may fail for some values of x in the domain of f.

Example

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- $= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) + 4 - [x^2 + 2x + 4]}{h}$

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- $f'(x) = 2x + 2.$

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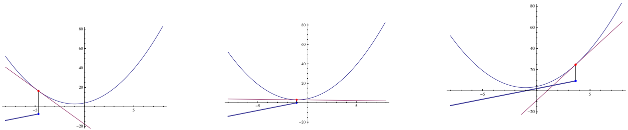
- $f'(x) = 2x + 2.$
- Since the domain of f is all real numbers and the above limit exists for all real numbers, the domain of f' is also all real numbers.

Graph of the derivative $f'(x)$

Below we see how the graph of $f(x) = x^2 + 2x + 4$ is related to the graph of its derivative $f'(x) = 2x + 2$, which gives the slope of the tangents to the graph of $f(x) = x^2 + 2x + 4$. (See Mathematica File)



Fill in $<, >$ *or* $=$ as appropriate:
When $f(x)$ is decreasing the function $f'(x)$___0
When $f(x)$ is increasing the function $f'(x)$___0
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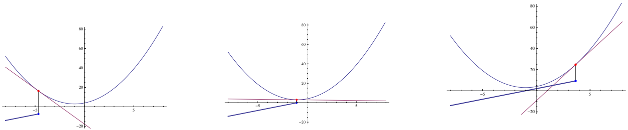

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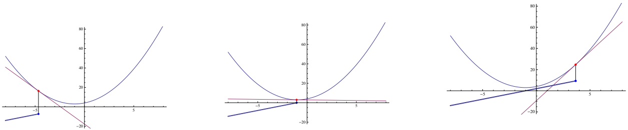

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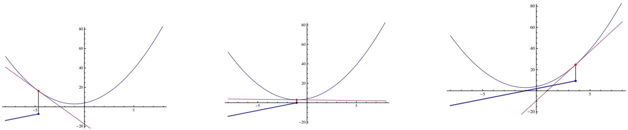

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- When $f(x)$ is decreasing the function $f'(x) < 0$
- When $f(x)$ is increasing the function $f'(x) > 0$
- At the turning point $x = -1$, $f'(x) = 0$

Domain of Derivative of $|x|$

Consider the function $f(x) = |x|$.
Does $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x > 0$?

Does $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x < 0$?

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What is the domain of $f'(x)$?

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What is the domain of $f'(x)$?

- Domain $f'(x) =$ all real numbers except 0.

Different Notation

Using $y = f(x)$, to denote that the independent variable is y, there are a number of notations used to denote the derivative of $f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

The symbols D and $\frac{d}{dx}$ are called differential operators, because when they are applied to a function, they transform the function to its derivative. The symbol $\frac{dy}{dx}$ should not be interpreted as a quotient rather it is a limit originating from the notation

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

When we evaluate the derivative at a number a, we use the following notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}.$$

Differentiability

Definition When a function f is defined in an open interval containing a, we say a function f is **differentiable** at a if $f'(a)$ exists. [That is, conditions 1, 2 and 3 from page 1 must be satisfied for $x = a$.] It is **differentiable on an open interval** , (a, b) (or (a, ∞) or $(-\infty, a)$) if it is differentiable at every number in the interval.

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Example Let $f(x) = |x|$. Is $f(x)$ differentiable at 0?

If $f(x)$ differentiable on the intervals $(-\infty, 0)$ and $(0, \infty)$.

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Is $f(x)$ continuous at 0?

- yes $f(x) = |x|$ is continuous at 0, because $\lim_{x \to 0} |x| = 0$. However, as we showed above, it is not differentiable at 0. (geometrically: there is a sharp point on the curve and no tangent line exists).

Differentiable at a implies continuous at a

Theorem If f is differentiable at a, then f is continuous at a.

In particular the theorem shows that if a function has a discontinuity at a point a, then it cannot be differentiable at a. (Note by the previous example, the converse is not true; a function can be continuous at a, but not differentiable at a).

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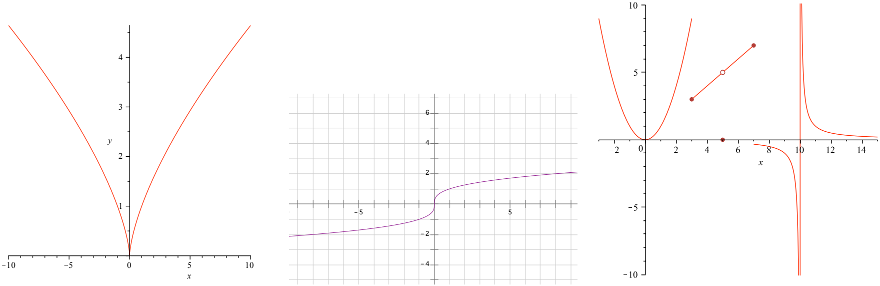
- The function might be continuous but the tangent line may be vertical, i.e. $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \pm\infty$.

Points where function is not differentiable

Example Identify the points in the graphs below where the functions are not differentiable.



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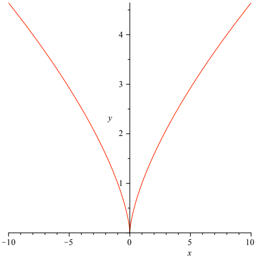

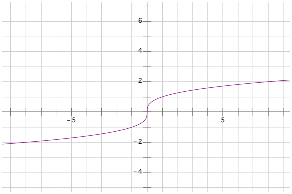

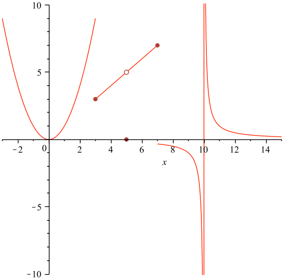

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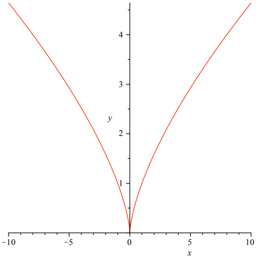

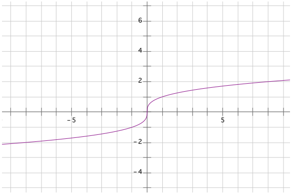

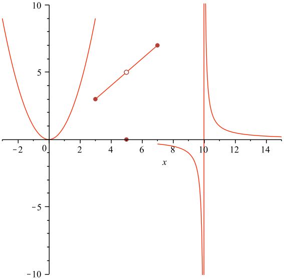

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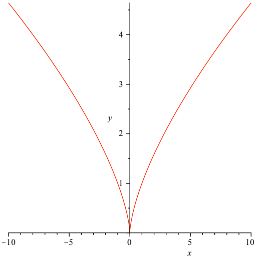

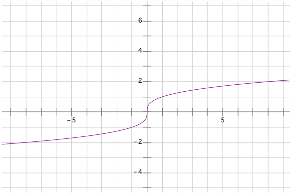

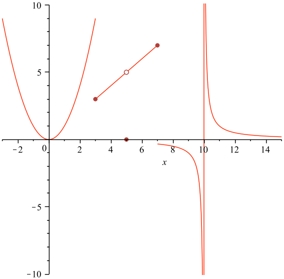

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- The graph in the center has a vertical tangent at $x = 0$. So the function is not differentiable at $x = 0$.
- The graph on the right is not continuous at $x = 3, x = 5, x = 7$ and $x = 10$. So the function cannot be differentiable at those points.

Higher Derivatives

We have seen that given a function $f(x)$, we can define a new function $f'(x)$. We can continue this process by defining a new function,

$$f''(x) = \frac{d}{dx} f'(x).$$

This is the second derivative of the function $f(x)$. This function gives the slope of the tangent to the curve $y = f'(x)$ at each value of x.
We can then define the third derivative of $f(x)$ as the derivative of the second derivative, etc...

Example Let $f(x) = x^2 + 2x + 4$. We saw above that the derivative of $f(x)$ is $f'(x) = 2x + 2$. Find and interpret the second derivative of $f(x)$;

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Acceleration

The second derivative gives us the rate of change of the rate of change. In the case of a position function $s = s(t)$ of an object moving in a straight line, the derivative $v(t) = s'(t)$ gives us the velocity of the moving object at time t and the second derivative $a(t) = v'(t) = s''(t)$ gives us the **acceleration** of the moving object at time t. This is the rate of change of the velocity at time t.

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- The acceleration is constant and $a(5) = 2$.

Notation for Higher derivatives

The second derivative is also denoted by

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