Name: ______________________________

Instructor: __________________________

Math 10550, Exam II
October 18, 2013

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hr. and 15 m..
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 12 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
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Multiple Choice	______________
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Multiple Choice

1.(6 pts.) A particle is moving along an axis. Its position at time t (seconds) is given by

$$s(t) = t^3 - 6t^2 + 9t,$$

where $s(t)$ is measured in feet. What is the total distance travelled by the particle between $t = 0$ and $t = 2$ seconds.

(a) 6 feet (b) 10 feet (c) 2 feet

(d) 4 feet (e) 5 feet

We calculate where the partical changes direction by taking the derivative of the position and setting it equal to zero.

$$s'(t) = 3t^2 - 12t + 9 = 0 \Longrightarrow t^2 - 4t + 3 = 0 \Longrightarrow t = 3, t = 1.$$

Therefore, the partical is moving towards the right from $t = 0$ until $t = 1$, and then to the left from $t = 1$ to $t = 2$. The distance traveled is

$$D = |s(1) - s(0)| + |s(2) - s(1)| = |1 - 6 + 9| + |8 - 24 + 18 - (1 - 6 + 9)| = 4 + 2 = 6.$$

2.(6 pts.) The height of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the height is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

(a) $190\,\text{cm}^2/\text{s}$ (b) $211\,\text{cm}^2/\text{s}$ (c) $140\,\text{cm}^2/\text{s}$

(d) $11\,\text{cm}^2/\text{s}$ (e) $24\,\text{cm}^2/\text{s}$

Area, A, of a rectangle satisfies

$$A = HW,$$

where H is height and W is width. Therefore, by the product rule

$$\frac{d}{dt}A = \frac{dH}{dt}W + H\frac{dW}{dt}.$$

Using $\frac{dH}{dt} = 8$, $\frac{dW}{dt} = 3$, $H = 20$, and $W = 10$ we have

$$\frac{d}{dt}A = 8 \times 10 + 20 \times 3 = 140cm^2/s.$$

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3.(6 pts.) Use linear approximation of $f(x) = \dfrac{1}{\sqrt{x}}$ at $a = 4$ to estimate $\dfrac{1}{\sqrt{3.9}}$.

(a) $\dfrac{1}{\sqrt{3.9}} \approx \dfrac{79}{160}$ (b) $\dfrac{1}{\sqrt{3.9}} \approx \dfrac{11}{20}$ (c) $\dfrac{1}{\sqrt{3.9}} \approx \dfrac{1}{2}$

(d) $\dfrac{1}{\sqrt{3.9}} \approx \dfrac{9}{20}$ (e) $\dfrac{1}{\sqrt{3.9}} \approx \dfrac{81}{160}$

The linear approximation

$$L(x) = f(a) + f'(a)(x - a)$$

for $f(x) = x^{-1/2} \Longrightarrow f'(x) = -\frac{1}{2}x^{-3/2}$ at $a = 4$ is

$$L(x) = \frac{1}{2} - \frac{1}{16}(x - 4).$$

We want to approximate $f(3.9)$ by

$$f(3.9) \approx L(3.9) = \frac{1}{2} - \frac{1}{16}(3.9 - 4) = \frac{1}{2} - \frac{1}{16}\left(-\frac{1}{10}\right) = \frac{1}{2} + \frac{1}{160} = \frac{80}{160} + \frac{1}{160} = \frac{81}{160}.$$

4.(6 pts.) Find the linearization $L(x)$ of the function $f(x) = \sin(2x)$ at $a = \dfrac{\pi}{4}$.

(a) $L(x) = 1 - \dfrac{\sqrt{2}\pi}{4} + \sqrt{2}x$ (b) $L(x) = 1$ (c) $L(x) = 1 - \dfrac{\pi}{2} + 2x$

(d) $L(x) = 1 + x$ (e) $L(x) = 1 + \dfrac{\pi}{2} - 2x$

The linear approximation

$$L(x) = f(a) + f'(a)(x - a)$$

for $f(x) = \sin(2x) \Longrightarrow f'(x) = 2\cos(2x)$ at $a = \frac{\pi}{4}$ is

$$L(x) = \sin(\frac{\pi}{2}) + 2\cos(\frac{\pi}{2})(x - \frac{\pi}{4}) = 1.$$

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5.(6 pts.) Find all critical points (critical numbers) of

$$f(x) = x^4 + \frac{16}{3}x^3 - 10x^2 - 12.$$

(a) $x = -2, 0, 2$ (b) $x = 5, 0, -1$ (c) $x = -5, 1$

(d) $x = 0, -2$ (e) $x = -5, 0, 1$

Since this is a polynomial, to find all of the critical points, we set the first derivative equal to zero.

$$f'(x) = 4x^3 + 16x^2 - 20x = 0.$$

Factoring yields

$$4x(x^2 + 4x - 5) = 0 \Longrightarrow 4x(x + 5)(x - 1) = 0.$$

This yields the critical points $x = -5$, $x = 0$ and $x = 1$.

6.(6 pts.) Let

$$f(x) = x^3 + 3x^2 - 24x.$$

Find the absolute maximum and absolute minimum values of f on the interval $[0, 10]$.

(a) Max at $x = 4$; Min at $x = 0$. (b) Max at $x = 10$; Min at $x = 0$.

(c) Max at $x = 4$; Min at $x = 1$. (d) Max at $x = 10$; Min at $x = 2$.

(e) Max at $x = 8$; Min at $x = 2$.

The absolute maximum and minimum occur weither at the endpoints or at a critical point in the interval. We take the first derivative to find critical points

$$f'(x) = 3x^2 + 6x - 24 = 0 \Longrightarrow x^2 + 2x - 8 = 0 \Longrightarrow (x + 4)(x - 2) = 0,$$

yields a critical point at $x = 2$ in the interval. Therefore, the maximum and minimum could be

$$f(0) = 0,\ f(2) = 8 + 12 - 48 = -28,\ f(10) = 1000 + 300 - 240 = 1060.$$

So the absolute minimum is at $x = 2$ and the absolute maximum is at $x = 10$.

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7.(6 pts.) Find the local maxima and minima of

$$f(x) = 3x^{2/3} - x$$

where $f(x)$ is defined for all real numbers x.

(a) f has a local minimum at $x = 0$ and a local maximum at $x = 8$.

(b) f has a local maximum at $x = 8$ and no local minimum.

(c) f has a local maximum at $x = 0$ and a local minimum at $x = 1/8$.

(d) f has a local minimum at $x = 0$ and a local maximum at $x = 1/8$.

(e) f has a local maximum at $x = 1/8$ and no local minimum.

Local maximum and minimum occur when the first derivative is zero, or does not exist. This is at

$$f'(x) = 2x^{-1/3} - 1 = 0 \Longrightarrow x = 8$$

and $f'(x)$ does not exist when $x = 0$. Since for x in a neighborhood of zero, $f(x)$ is positive, and $f(0) = 0$, we conclude $x = 0$ is a local minimum. To determine whether $x = 8$ is a local maximum or minimum, we examine how the first derivative changes as we pass through $x = 8$. Let's rewrite the derivative as $f'(x) = \dfrac{2 - \sqrt[3]{x}}{\sqrt[3]{x}}$. From this we can see that $f'(x)$ goes from positive to negative as we pass through $x = 8$. Thus $x = 8$ is a local maximum.

8.(6 pts.) Let

$$f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 10.$$

On which of the following intervals is the graph of the function f **both** decreasing and concave upward on the entire interval?

(a) $(-\infty, 2)$ (b) $(1, 2)$ (c) $\left(-\infty, \frac{3}{2}\right)$

(d) $\left(\frac{3}{2}, 2\right)$ (e) $(0, 2)$

The function is decreasing wherever the first derivative is negative:

$$f('x) = x^2 - 3x + 2 = 0 \Longrightarrow (x - 2)(x - 1) = 0.$$

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Note that the first derivative is a parabola which opens upward, so it must be negative on $(1, 2)$ and positive elsewhere. Therefore the function is decreasing on $(1, 2)$.

To determine where f is concave up, we take the second derivative, to find

$$f''(x) = 2x - 3 = 0 \Longrightarrow x = \frac{3}{2}.$$

Since $f''(2) > 0$, we conclude f is concave up on $(\frac{3}{2}, \infty)$. Conbining the two intervals, we have f is decreasing and concave up on $(\frac{3}{2}, 2)$.

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9.(6 pts.) Consider the function

$$f(x) = \frac{3x^3 - 3}{(2x+2)(x^2 - 7x + 10)}.$$

Which of the following is true?

(a) f has a horizontal asymptote at $y = 1$ and vertical asymptotes at $x = -1, 2, 5$.

(b) f has a horizontal asymptote at $y = \frac{3}{2}$ and vertical asymptotes at $x = 1, 2, 5$.

(c) f has a horizontal asymptote at $y = \frac{3}{2}$ and vertical asymptotes at $x = -1, 2, 5$.

(d) f has a horizontal asymptote at $y = -1$ and vertical asymptotes at $x = -1, 2, 5$.

(e) f has a horizontal asymptote at $y = \frac{3}{2}, -\frac{3}{2}$ and vertical asymptotes at $x = -1, 2, 5$.

By expanding the product in the denominator we see the function has the form

$$\frac{3x^3 \pm \text{lower order terms}}{2x^3 \pm \text{lower order terms}}$$

Thus there is a horizontal asymptote at $y = \frac{3}{2}$, since the long-term behavior of the function is determined by the highest order terms.

There will be a vertical asysmptote when the denominator is zero and the numerator is non-zero. This happens at $x = -1$ which makes $2x + 2$ zero. This also happens at $x = 2$ and $x = 5$, which both make the term $x^2 - 7x + 10 = (x-2)(x-5)$ to be zero.

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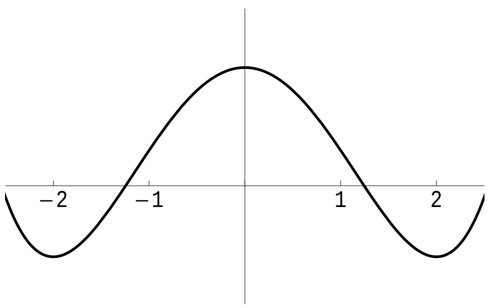
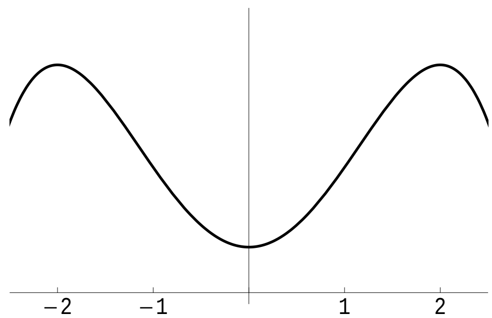
10.(6 pts.) Let f be a function of x. The table below shows whether the functions $f'(x)$ and $f''(x)$ are positive, negative or have value 0 at each of the given values of x.

x	-2	0	2
$f'(x)$	$=0$	$=0$	$=0$
$f''(x)$	>0	$=0$	<0

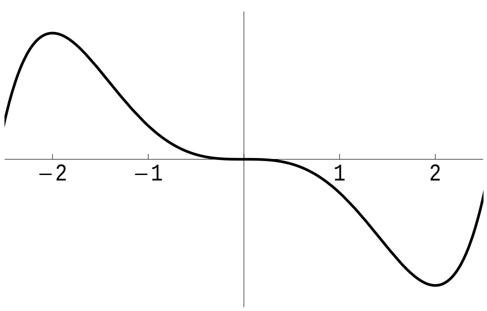
Which of the graphs shown below is a feasible graph of $f(x)$?
(**Note that the label for each graph is given on the lower left of the graph.**)



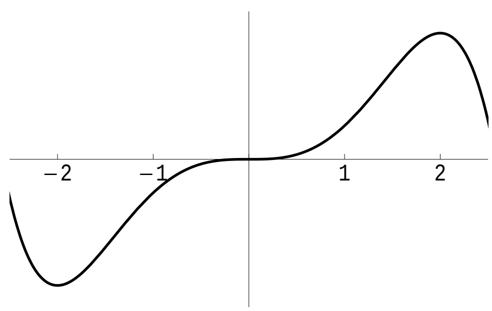
(a)



(b)



(c)



(d)

(e) None of the above

The second derivative values at -2 and 2 require that the graph is concave up at $x = -2$ and concave down at $x = 2$. (d) is the only graph satisfying these conditions. It has a horizontal tangent line at the points given in the table, so it seems like a good candidate. The only remaining thing to check is the second derivative at $x = 0$. The concavity of the graph changes from down to up at $x = 0$, so we have an inflection point and the second derivative is zero. Thus the answer is (d).

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(13 pts.) Show that

$$x^5 + 2x^3 + 2x - 3 = 0$$

has one and exactly one solution. Identify the theorem(s) you are using.

First note that $1^5 + 2 + 2 - 3 = 2$ and $(-1)^5 + 2(-1)^3 + 2(-1) - 3 = -8$. Moreover, the polynomial is continuous on the whole real line and therefore, by the Intermediate Value Theorem, there exists at least one zero of the polynomial in the interval $(-1, 1)$.

Next we show there is only one zero. We have for

$$f(x) = x^5 + 2x^3 + 2x - 3,$$

$$f'(x) = 5x^4 + 6x^2 + 2 \geq 2 > 0.$$

Hence the polynomial is monotonically increasing, and therefore cannot cross the x axis a second time.

We can also show that the polynomial has only one zero by using Rolle's Theorem. The argument above showed that there is a zero in $(-1, 1)$. Let's say this zero is at $x = a$. Suppose there were a second zero at $x = b$, where b is some other point. Then Rolle's Theorem says that there must be a point c between a and b such that the derivative at c is zero : $f'(c) = 0$. But this is impossible, since the derivative $f'(x) = 5x^4 + 6x^2 + 2$ is always strictly greater than zero. Therefore, there must not be a second solution. We may apply Rolle's Theorem in this case because the polynomial $f(x)$ is continuous and differentiable on the entire real line.

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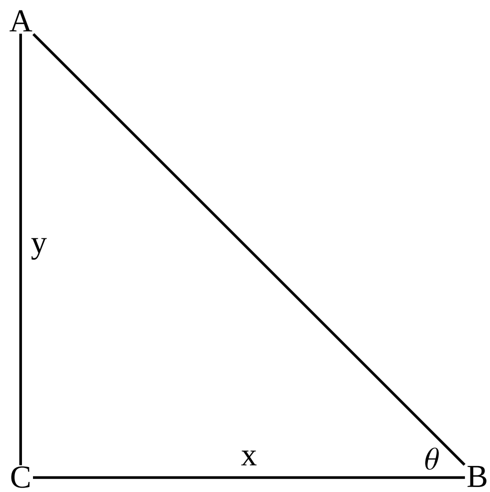
12.(13 pts.) Car A and car B are approaching the intersection "C" of two streets intersecting at a right angle. Car A is going South at 45 mph, car B is heading West at 30 mph. We denote the angle $\angle(C, B, A)$ by θ (measured in radians), the distance from C to B by x, and the distance from C to A by y. At what rate is the angle θ changing when car A and car B are both 1 mile from the intersection?



First note that $\frac{dx}{dt} = -30$ mph and $\frac{dy}{dt} = -45$ mph.

Now we use the equation $\tan(\theta) = \frac{y}{x}$. Via implicit differentiation we obtain

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

When $x = y = 1$, the hypoteneuse has length $\sqrt{2}$, so that $\sec^2(\theta) = \left(\frac{\sqrt{2}}{1}\right)^2 = 2$. Substituting this value as well as the known values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and $x = y = 1$ into the equation above, we get

$$2\frac{d\theta}{dt} = \frac{1(-45) - 1(-30)}{1^2}$$

which simplifies to $\frac{d\theta}{dt} = -\frac{15}{2}$ rad/hr.

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13.(14 pts.) Suppose $f(x)$ is a function which is continuous and differentiable on the interval $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$ with

$$f'(x) = 1 - \sin^2 x.$$

Warning: the formula shown above is for the DERIVATIVE of $f(x)$

(a) Find all critical points (critical numbers) of the function $f(x)$ in the given interval.

$f(x)$ has a critical point whenever the derivative is zero. This happens if $1-\sin^2 x = 0$, so that $\sin x = \pm 1$. On the given interval $\sin x$ has absolute value 1 if $x = \pm\frac{\pi}{2}$. These are the two critical points.

(b) List the subintervals of $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$ where f is increasing / decreasing.

Since $-1 \leq \sin x \leq 1$, we know that the derivative $f'(x)$ is always greater than or equal to zero. Thus the function is increasing on the whole interval $(-\frac{3\pi}{4}, \frac{3\pi}{4})$.

(c) List all local maxima and local minima of f in the interval $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$, or say so if there are none.

The sign of the derivative never changes, as we noted above in part (b). So there are no local maxima or minima.

(d) List the subintervals of $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$ where f is concave up / concave down.

$f''(x) = -2\sin x \cos x$. This is zero whenever sine or cosine are zero on the given interval. This happens at $x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$. We then check points to find the sign of the second derivative on the different intervals. We will use what we know about the sign of $\sin x$ and $\cos x$ to find the sign of the second derivative without computing specific values.

$f''(x) = -2(\text{negative})(\text{negative}) = \text{negative}$ for $-\frac{3\pi}{4} < x < -\frac{\pi}{2}$ since both the sine and cosine are negative in that region. Similar analysis of sine and cosine on the other regions give us (note that the first sign goes with $\sin x$ and the second with $\cos x$)

$f''(x) = -2(\text{negative})(\text{positive}) = (\text{positive})$ for $-\frac{\pi}{2} < x < 0$

$f''(x) = -2(\text{positive})(\text{positive}) = (\text{negative})$ for $0 < x < \frac{\pi}{2}$

$f''(x) = -2(\text{positive})(\text{negative}) = (\text{positive})$ for $\frac{\pi}{2} < x < \frac{3\pi}{4}$

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So we find that $f(x)$ is concave up on $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \frac{3\pi}{4})$, and concave down on $(-\frac{3\pi}{4}, -\frac{\pi}{2})$ and $(0, \frac{\pi}{2})$

(e) List all inflection points of f in the interval $\left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right)$, or say so if there are none.

The points of inflection are found using what we did in part (d). The sign of the second derivative changes at each of the three points we found earlier, so the inflection points are $x = -\frac{\pi}{2}, 0,$ and $\frac{\pi}{2}$.

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