Math 10550, Exam III
November 15, 2011

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

**PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!**

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Multiple Choice ____________

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Multiple Choice

1. (6 pts.) Estimate the area under the graph of \( y = f(x) \) between \( x = 0.5 \) and \( x = 2.0 \), using a Riemann sum with three equal subintervals, using the left-hand endpoints.

The left end point approximation with \( n = 3 \) is the area of the shaded region above, which is \( 6 \times (0.5)^2 = 1.25 \).

(a) 3.0  (b) 2.0  (c) 1.5  (d) 1.0  (e) 4.0

2. (6 pts.) Find \( \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} \).

\[
\lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6} \lim_{n \to \infty} \frac{(n+1)(2n+1)}{n^2} = \frac{1}{6} \lim_{n \to \infty} \frac{n+1}{n} \lim_{n \to \infty} \frac{(2n+1)}{n} = \frac{2}{6} = \frac{1}{3}.
\]

(a) \( \infty \)  (b) 1  (c) \( 1/6 \)  (d) 0  (e) \( 1/3 \)
3. (6 pts.) Find the equation of the slant asymptote as \( x \to \infty \) of the function
\[
f(x) = \frac{2x^3}{x^2 - 1}.
\]
Using long division, we see that
\[
\frac{2x^3}{x^2 - 1} = 2x + \frac{2x}{x^2 - 1}.
\]
Therefore
\[
\lim_{x \to \infty} [f(x) - 2x] = \lim_{x \to \infty} \frac{2x}{x^2 - 1} = 0.
\]
Therefore \( y = 2x \) is a slant asymptote to the graph of the function \( f(x) \).

(a) \( y = 0 \)  \hspace{1cm} (b) \( y = 2x \)  \hspace{1cm} (c) \( y = -x + 2 \)

(d) \( y = -2x \)  \hspace{1cm} (e) \( y = 2x + 1 \)

4. (6 pts.) In finding an approximate solution to the equation \( x^4 - 2x^3 - 5 = 0 \) using Newton’s method with initial approximation \( x_1 = 2 \), what is \( x_2 \)?

We have
\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)}.
\]
\( f'(x) = 4x^3 - 6x^2 \) and \( f'(2) = 4(2)^3 - 6(2)^2 = 32 - 24 = 8. \)
Also \( f(2) = 2^4 - 2(2^3) - 5 = 16 - 16 - 5 = -5. \)
Therefore
\[
x_2 = 2 - \frac{-5}{8} = 2 + \frac{5}{8} = \frac{21}{8}.
\]

(a) \( 5/8 \)  \hspace{1cm} (b) \( 18/5 \)  \hspace{1cm} (c) \( 2/5 \)  \hspace{1cm} (d) \( 21/8 \)  \hspace{1cm} (e) \( 11/8 \)
5. (6 pts.) The graph of the function $v(t)$ is given below:

Find $\int_0^4 v(t) \, dt$.

The definite integral $\int_0^4 v(t) \, dt$ is the net signed area of the graph between 0 and 4. From the picture, we can see that the net signed area is $-(\text{area of rectangle shown above}) = -4$.

(a) $-4$  (b) $0$  (c) $4$  (d) $-2$  (e) $2$

6. (6 pts.) Find an antiderivative $F(x)$ of $f(x) = 2x + 3\sqrt{x}$ satisfying $F(1) = 4$. Which of the following is $F(4)$?

$$F(x) = \frac{2x^2}{2} + 3\frac{x^{3/2}}{3/2} + C = x^2 + 2x^{3/2} + C.$$  

$F(1) = 4$ implies that $1^2 + 2(1)^{3/2} + C = 4$ which implies that $1 + 2 + C = 4$, which implies that $C = 1$.

Therefore $F(x) = x^2 + 2x^{3/2} + 1$ and $F(4) = 4^2 + 2(4)^{3/2} + 1 = 16 + 2(8) + 1 = 33$.

(a) $16$  (b) $9$  (c) $27$  (d) $33$  (e) $7$
7. (6 pts.) Evaluate the Riemann sum for \( f(x) = 2 - x^2 \) with \( 0 \leq x \leq 2 \), using four subintervals and taking the sample points to be the right-hand endpoints of the intervals.

We use 4 subintervals, with length \( \Delta x = \frac{2 - 0}{4} = 1/2 \) and endpoints

\[
x_0 = 0, \quad x_1 = 1/2, \quad x_2 = 1, \quad x_3 = 3/2, \quad x_4 = 2.
\]

We make a table of values of function values at the endpoints:

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<th>( x_i )</th>
<th>( f(x_i) = 2 - 2x_i^2 )</th>
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<td>0</td>
<td>2</td>
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<td>1/2</td>
<td>7/4</td>
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<td>1</td>
<td>-1/4</td>
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The value of the Riemann sum using the right end points with \( n = 4 \) is

\[
R_4 = \Delta x [f(x_1) + f(x_2) + f(x_3) + f(x_4)]
\]

\[
\]

(a) \quad 0.2 \quad (b) \quad 1.5 \quad (c) \quad 2.5 \quad (d) \quad 0.25 \quad (e) \quad 0.36

8. (6 pts.) By interpreting the integral as an area, evaluate \( \int_{-2}^{2} (4 - x^2)^{1/2} \, dx \).

The graph of \( y = (4 - x^2)^{1/2} \) is a semicircle of radius 2 as shown below. (since squaring both sides gives \( y^2 = 4 - x^2 \) and \( (4 - x^2)^{1/2} > 0 \).

The value of the definite integral \( \int_{-2}^{2} (4 - x^2)^{1/2} \, dx \) is the area under this curve between -2 and 2. The area under the curve between -2 and 2 is one half of the area of a circle with radius 2. Therefore we get

\[
\int_{-2}^{2} (4 - x^2)^{1/2} \, dx = \frac{1}{2} \pi 4 = 2\pi.
\]

(a) \quad \frac{\sqrt{2}}{2} \pi \quad (b) \quad 2\pi \quad (c) \quad 4\pi \quad (d) \quad \pi \quad (e) \quad 0
9. (6 pts.) Evaluate \( \int_{1}^{2} \frac{x^2 + \sqrt{x}}{x} \, dx \).

\[
\int_{1}^{2} \frac{x^2 + \sqrt{x}}{x} \, dx = \int_{1}^{2} x + x^{-1/2} \, dx = \left. \frac{x^2}{2} + x^{1/2} \right|_{1}^{2} = \frac{4}{2} + 2\sqrt{2} - \frac{1}{2} + 2 = 2\sqrt{2} - \frac{1}{2}.
\]

(a) \( 2 - \frac{\sqrt{2}}{2} \) \hspace{1cm} (b) \( 3\sqrt{2} \) \hspace{1cm} (c) \( 2\sqrt{2} + \frac{1}{2} \)

(d) \( \sqrt{2} + 2 \) \hspace{1cm} (e) \( 2\sqrt{2} - \frac{1}{2} \)

10. (6 pts.) If \( F(x) = \int_{x^2}^{4} (t + 1) \, dt \), find \( F'(x) \).

We use the chain rule:

\[
F'(x) = \frac{d}{dx} \int_{x^2}^{4} (t + 1) \, dt = \frac{d}{d(x^2)} \int_{x^2}^{4} (t + 1) \, dt \frac{dx^2}{dx} = -(x^2 + 1)2x = -2x^2 + 1.
\]

(a) \( -2x^3 - 2x \) \hspace{1cm} (b) \( 2x^3 + 2x \) \hspace{1cm} (c) \( x^2 + 1 \)

(d) \( x \) \hspace{1cm} (e) \( \frac{x^2}{2} + x \)


Partial Credit

You must show your work on the partial credit problems to receive credit!

11. (10 pts.) A rectangle $CODP$ (with sides $CP$ and $OD$ parallel to the $x$-axis) is inscribed in the region bounded by the lines $x + y = 4$ and the coordinate axes, with the corner $P$ being on the line segment $AB$ (including possibly at $A$ or at $B$).

(a) Write the area $A(x)$ of the rectangle in terms of $x$, the $x$-coordinate of $P$.

We have $|OD| = x$ and $|DP| = 4 - x$, where $|OD|$ and $|DP|$ denote the length of the corresponding line segments. Therefore $A(x) = x(4 - x) = 4x - x^2$.

(b) What is the range of possible values of $x$?

Since we allow $A$ and $B$ as corners of the rectangle, $x$ may take any value from the interval $[0, 4]$.

(c) Find the value of $x$ that maximizes the area $A(x)$. For full credit, you must show that your answer maximizes $A(x)$.

We must find the maximum of the continuous function $A(x)$ on the closed interval $[0, 4]$. We know that the maximum occurs either at a critical point or at the end points.

Critical Points: $A'(x) = 4 - 2x$. We have a critical point where $4 - 2x = 0$ or $x = 2$.

To verify that we have a maximum of the function $A(x)$ at $x = 2$, we compare function values:

$A(0) = 0$, $A(2) = 4$, $A(4) = 0$.

By comparison, we see that the area is maximized when $x = 2$. 


12. (10 pts.) A ship is sailing along the path \( y = 3x + 1 \) (with units being nautical miles). A lighthouse is located at the point \( (1, 1) \). How close does the ship come to the lighthouse? For full credit, be sure to show why the answer you have found is the **minimum** distance.

**Hint:** It might be easier to first minimize the **square** of the distance from the ship to the lighthouse.

A point on the ship’s path has co-ordinates \( (x, y) = (x, 3x + 1) \). The distance from such a point to the point \( (1, 1) \) is given by

\[
D(x) = \sqrt{(x - 1)^2 + (3x + 1 - 1)^2} \\
= \sqrt{(x - 1)^2 + (3x)^2} \\
= \sqrt{x^2 - 2x + 1 + 9x^2} \\
= \sqrt{10x^2 - 2x + 1}
\]

To find the minimum of \( D(x) \), we look at the critical points of the function.

\[
D'(x) = \frac{1}{2} \cdot \frac{20x - 2}{\sqrt{10x^2 - 2x + 1}} = \frac{20x - 2}{2\sqrt{10x^2 - 2x + 1}}.
\]

The critical points occur when \( D'(x) = 0 \) or \( D'(x) \) does not exist.

Since \( 10x^2 - 2x + 1 = (x - 1)^2 + (3x + 1 - 1)^2 \) has no zeros (because the point \( (1, 1) \) is not on the curve \( y = 3x + 1 \)), we can conclude that \( D'(x) \) exists everywhere.

\[
D'(x) = 0 \text{ if } 20x - 2 = 0 \text{ or } \boxed{x = 1/10}.
\]

Since

\[
D'(x) = \frac{20(x - 1/10)}{2\sqrt{10x^2 - 2x + 1}}
\]

we have \( D'(x) > 0 \) if \( x > 1/10 \) and \( D'(x) < 0 \) if \( x < 1/10 \). This tells us that \( D(x) \) has a global minimum at \( x = 1/10 \).

When \( x = 1/10 \), the distance between the ship and the lighthouse is

\[
D(1/10) = \sqrt{10/100 - 2/10 + 1} = \sqrt{90/100} = \frac{3\sqrt{10}}{10}.
\]
13. (10 pts.) Let \( f(x) = \frac{x\sqrt{x^2 + 1}}{x^2 - 1} \).

(a) Find the equations of all horizontal asymptotes of \( y = f(x) \).

\[
\lim_{x \to \infty} \frac{x\sqrt{x^2 + 1}}{x^2 - 1} = \lim_{x \to \infty} \frac{(x\sqrt{x^2 + 1})/x^2}{(x^2 - 1)/x^2} = \lim_{x \to \infty} \frac{\sqrt{x^2 + 1}/x}{1 - 1/x^2} = \lim_{x \to \infty} \frac{\sqrt{1 + 1/x^2}}{1 - 1/x^2} = 1
\]

\[
\lim_{x \to -\infty} \frac{x\sqrt{x^2 + 1}}{x^2 - 1} = \lim_{x \to -\infty} \frac{(x\sqrt{x^2 + 1})/x^2}{(x^2 - 1)/x^2} = \lim_{x \to -\infty} \frac{\sqrt{x^2 + 1}/x}{1 - 1/x^2} = \lim_{x \to -\infty} \frac{-\sqrt{1 + 1/x^2}}{1 - 1/x^2} = -1
\]

Therefore the lines \( y = 1 \) and \( y = -1 \) are horizontal asymptotes to the graph of \( y = f(x) \).

(b) Find the equations of all vertical asymptotes of \( y = f(x) \).

The lines \( x = 1 \) and \( x = -1 \) are vertical asymptotes to the graph of \( y = f(x) \), since

\[
\lim_{x \to 1^+} \frac{x\sqrt{x^2 + 1}}{x^2 - 1} = \infty
\]

and

\[
\lim_{x \to (-1)^+} \frac{x\sqrt{x^2 + 1}}{x^2 - 1} = -\infty
\]
14. (10 pts.) A particle is moving along a vertical axis, with the upward direction positive. Its velocity at time \( t \geq 0 \) (measured in seconds) is \( v(t) = 8 - 6t \) (measured in meters per second). Its position at time \( t \) is \( s(t) \), with \( s(0) = 0 \).

(a) Find \( s(t) \). Find a time \( t > 0 \) for which \( s(t) = 0 \).

\[
v(t) = 8 - 6t \\
s(t) = 8t - \frac{6t^2}{2} + C = 8t - 3t^2 + C \\
s(0) = 0 \rightarrow C = 0 \\
S(t) = 0 \text{ if } 8t - 3t^2 = 0 \text{ if } t(8 - 3t) = 0 \text{ if } t = 0 \text{ or } t = 8/3.
\]

(b) At the time found in part (a), at what speed is the particle moving, and in what direction?

When \( t = 8/3 \), \( v(t) = 8 - \frac{6 \cdot 8}{3} = 8 - 16 = -8 \).

The speed at this time is 8 meters per second and the particle is moving downwards.

(c) Find the total distance that the particle travels between \( t = 0 \) and \( t = 1 \).

The total distance traveled is given by

\[
\int_0^1 |v(t)| \, dt = \int_0^1 |8 - 6t| \, dt
\]

Since \( 8 - 6t > 0 \) if \( 0 \leq t \leq 1 \), we have

\[
\int_0^1 |8 - 6t| \, dt = \int_0^1 (8 - 6t) \, dt = (8t - 3t^2)|_0^1 = (8 - 3) - 0 = 5 \text{ meters}
\]
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