MATH 10550, EXAM 1 SOLUTIONS

1. Evaluate the following limit

$$\lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{x^2}.$$

Solution.

$$\lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{x^2} = \lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{x^2} \cdot \frac{2+\sqrt{4-x^2}}{2+\sqrt{4-x^2}} = \lim_{x\to 0} \frac{4-(4-x^2)}{x^2(2+\sqrt{4-x^2})}$$

$$= \lim_{x\to 0} \frac{1}{2+\sqrt{4-x^2}} = \frac{1}{4}.$$

2. For which value of the constant c is the function $f(x)$ continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} c^2x - c & x \le 1 \\ cx - x & x > 1. \end{cases}$$

Solution. The partial functions of $f(x)$ are continuous for $x < 1$ and $x > 1$ because they are polynomials. To get $f(x)$ continuous on $(-\infty, \infty)$ we need

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x),$$

or at $x = 1$, $c^2x - c = cx - x$. This happens when $c^2 - c = c - 1$. Rearranging gives $0 = c^2 - 2c + 1 = (c-1)^2$ and $c = 1$.

3. Given that f and g are differentiable at $x = 3$ and that $f(3) = 2$, $g(3) = -1$, $f'(3) = -4$ and $g'(3) = 3$, what is $\left(\frac{f}{g}\right)'(3)$?

Solution. By quotient rule,

$$(\frac{f}{g})'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{(g(3))^2} = \frac{-4\cdot(-1) - 2\cdot 3}{(-1)^2} = -2.$$

4. For $f(x) = \sqrt[3]{x^5} + \frac{6}{\sqrt[5]{x^3}}$, find $f'(x)$.

Solution. $f(x) = x^{5/3} + 6x^{-3/5}$. Thus

$$f'(x) = \frac{5}{3}x^{2/3} + 6\left(-\frac{3}{5}x^{-8/5}\right) = \frac{5\sqrt[3]{x^2}}{3} - \frac{18}{5\sqrt[5]{x^8}}.$$

5. Find the equation of the tangent line to $y = \sqrt{x^2 - 1}$ at the point $(2, \sqrt{3})$.

Solution. $\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}}$ by the chain rule.

Then at $(2, \sqrt{3})$ the derivative of y is $\frac{2}{\sqrt{3}}$. This gives us the slope of the tangent line. So the equation of the tangent line at $(2, \sqrt{3})$ is given by

$$y - \sqrt{3} = \frac{2}{\sqrt{3}}(x - 2).$$

By simplifying, we have

$$y = \frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}.$$

6. Compute

$$\lim_{x \to \pi/2+} \tan x.$$

Solution. From the graph of $y = \tan x$,

$$\lim_{x \to \pi/2+} \tan x = -\infty.$$

7. Find the derivative of

$$f(x) = x^2 \cos(\sqrt{x^3 - 1} + 2).$$

Solution.

$$\begin{aligned}
f'(x) &= 2x\cos(\sqrt{x^3-1}+2) + x^2\frac{d}{dx}\cos(\sqrt{x^3-1}+2) \text{ (Product Rule)} \\
&= 2x\cos(\sqrt{x^3-1}+2) - x^2\sin(\sqrt{x^3-1}+2)\frac{d}{dx}(\sqrt{x^3-1}+2) \text{ (Chain Rule)} \\
&= 2x\cos(\sqrt{x^3-1}+2) - \frac{x^2}{2\sqrt{x^3-1}}\sin(\sqrt{x^3-1}+2)\frac{d}{dx}(x^3-1) \text{ (Chain Rule)} \\
&= 2x\cos(\sqrt{x^3-1}+2) - \frac{3x^4}{2\sqrt{x^3-1}}\sin(\sqrt{x^3-1}+2).
\end{aligned}$$

8. If $f(x) = x^2 \cos x$, find $f''(x)$.

Solution. Using Product Rule, we get

$$\begin{aligned}
f'(x) &= 2x\cos x - x^2\sin x, \\
\text{and} \quad f''(x) &= 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x \\
&= 2\cos x - 4x\sin x - x^2\cos x.
\end{aligned}$$

9. A ball is thrown straight upward from the ground with the initial velocity $v_0 = 96$ft/s. Find the highest point reached by the ball. Hint: The height of the ball at time t is given by $y(t) = -16t^2 + 96t$.
Solution. Velocity of the ball at time t is given by

$$v(t) = y'(t) = -32t + 96.$$

The ball reaches the highest point when $v(t) = 0$, i.e. when $t = 3$ seconds, so the height of the ball at 3 seconds is

$$\begin{aligned} y(3) &= -16(3)^2 + 96(3) \text{ ft.} \\ &= -144 + 288 \text{ ft.} \\ &= 144 \text{ ft.} \end{aligned}$$

10. Find the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}.$$

Solution.

$$\begin{aligned} \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} &= \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \cdot \frac{1 + \cos x}{1 + \cos x} \\ &= \lim_{x \to 0} \frac{1 - \cos^2 x}{x \sin x \cdot (1 + \cos x)} \\ &= \lim_{x \to 0} \frac{\sin^2 x}{x \sin x \cdot (1 + \cos x)} \\ &= \lim_{x \to 0} \frac{\sin x}{x \cdot (1 + \cos x)} \\ &= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{1 + \cos x} \\ &= 1 \cdot \frac{1}{2} = \frac{1}{2} \end{aligned}$$

11. Find the equation of the tangent line to the curve $y = \frac{x^3}{3} - x^2 + 1$ which is parallel to the line $y + x = 4$.

Solution. The line parallel to the line $y + x = 2$ will have the same slope, namely -1. So we need to find the point on the curve which has slope -1. $y' = x^2 - 2x$. We solve for x given $y' = -1$:

$$x^2 - 2x = -1 \Longrightarrow (x-1)(x-1) = 0 \Longrightarrow x = 1.$$

Plugging into the equation for the curve we see that $y = 1/3$ at this point. The tangent line at $(1, \frac{1}{3})$ is given by

$$y - \frac{1}{3} = -(x-1),$$

or

$$y = -x + \frac{4}{3}.$$

12. Show that there are at least *two* roots of the equation

$$x^4 + 6x - 2 = 0.$$

Justify your answer and identify the theorem you use.

Solution. Let $f(x) = x^4 + 6x - 2$. Then $f(-2) = 2$, $f(0) = -2$ and $f(1) = 5$. Since $f(x)$ is a polynomial, f is continuous on the real line. We have $f(-2) > 0 > f(0)$. So, by the **Intermediate Value Theorem**, there exists a number c between -2 and 0 such that $f(c) = 0$. Similarly, there exists a number d between 0 and 1 such that $f(d) = 0$.

Note: The choices $x = -2, 0, 1$ are not the only possibilities.

13. Given

$$y = \frac{1}{x^2 + 1},$$

find y' using the **definition** of the derivative.

Solution.

$$
\begin{aligned}
\text{Let } f(x) &= \frac{1}{x^2+1}. \\
\text{Then } y' = f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2+1} - \frac{1}{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(x^2+1)-((x+h)^2+1)}{((x+h)^2+1)\cdot(x^2+1)} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2}+\cancel{1}-\cancel{x^2}-2xh-h^2-\cancel{1}}{h((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}((x+h)^2+1)(x^2+1)} \\
&= \lim_{h\to 0} \frac{-2x-h}{((x+h)^2+1)(x^2+1)} \\
&= \frac{-2x-0}{((x+0)^2+1)(x^2+1)} \\
&= -\frac{2x}{(x^2+1)^2}.
\end{aligned}
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