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Exam 3 Solutions

Multiple Choice

1.(6 pts.) Find the equation of the slant asymptote to the function

$$f(x) = \frac{3x^3 + 2x^2 + 5x + 2}{x^2 + 1}$$

Solution:
We have

$$\begin{array}{r|l} & 3x + 2 \\ \hline x^2 + 1 & 3x^3 + 2x^2 + 5x + 2 \\ & -3x^3 \qquad\quad - 3x \\ \hline & 2x^2 + 2x + 2 \\ & -2x^2 \qquad - 2 \\ \hline & 2x \end{array}$$

so the slant asymptote is $y = 3x + 2$.

2.(6 pts.) Calculate the indefinite integral

$$\int \frac{x + \sqrt[5]{x}}{x}\, dx$$

Solution:

$$\begin{aligned} \int \frac{x + \sqrt[5]{x}}{x} dx &= \int \left(\frac{x}{x} + \frac{\sqrt[5]{x}}{x} \right) dx \\ &= \int (1 + x^{-4/5}) dx \\ &= x + 5x^{1/5} + C \\ &= x + 5\sqrt[5]{x} + C. \end{aligned}$$

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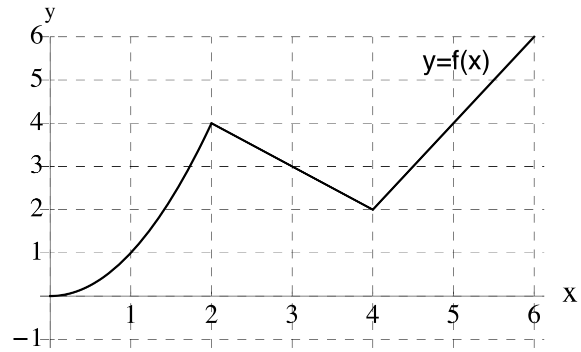
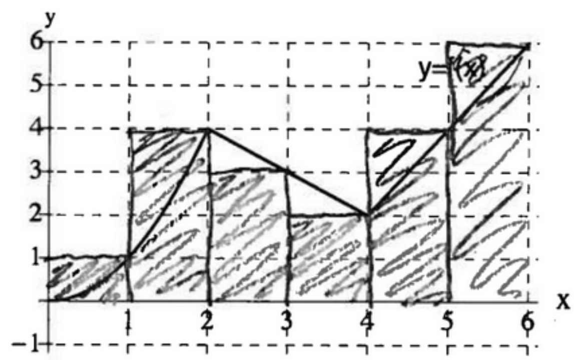
3.(6 pts.) Estimate the area under the graph of $y = f(x)$ between $x = 0$ and $x = 6$ using the Riemann sum which gives the right end point approximation with 6 approximating rectangles whose bases are of equal length (i.e. use R_6).



Solution:
First, we draw our six rectangles with right endpoints.



Now, since $\Delta x = 1$, we have

$$R_6 = \sum_{i=1}^{6} f(i) \cdot 1 = (1 + 4 + 3 + 2 + 4 + 6) \cdot 1 = 20.$$

4.(6 pts.) In finding the approximate solution to

$$x^3 - 4$$

using Newton's method with initial approximation $x_1 = 1$, what is x_3?
Solutions:
Let $f(x) = x^3 - 4$, the formula to find x_2 is

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\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= x_1 - \frac{x_1^3 - 4}{3x_1^2} \\
&= 1 - \frac{-3}{3} \\
&= 2
\end{aligned}
$$

Then,

$$
\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= x_2 - \frac{x_2^3 - 4}{3x_2^2} \\
&= 2 - \frac{4}{12} \\
&= 2 - \frac{1}{3} \\
&= \frac{5}{3}
\end{aligned}
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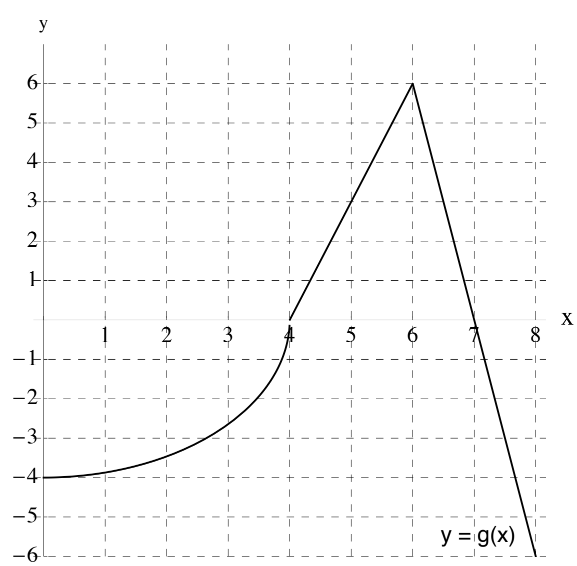
5.(6 pts.) A ball is thrown upwards from a height of 20 feet above the surface of the planet Minerva with an initial velocity of 6 feet per second (at time $t = 0$). The ball has a constant acceleration of -2 ft/sec^2. What is the maximum height (from the surface of the planet) reached by the ball?

Solution:

Let $h(t)$ denote the height of the ball, in feet, t seconds after it is thrown. We are told that the accelleration due to gravity on Minerva is $-2ft/s^2$, so we know $f''(t) = -2$. Antidifferentiating, we get that $f'(t) = -2t + c$, for some c. The initial velocity is $6ft/s$, so $f'(0) = -2 \cdot 0 + c = c = 6$, so $f'(t) = -2t + 6$. Antidifferentiating again, we obtain $f(t) = -t^2 + 6t + d$, for some d. The initial position is 20 feet above the surface, so $f(0) = d = 20$, and $f(t) = -t^2 + 6t + 20$.

We are asked the maximum height. The maximum height will be attained when the ball has zero velocity, i.e. when $f'(t) = -2t + 6 = 0$. This occurs when $t = 3$. At $t = 3$, the height is $f(3) = -3^2 + 6 \cdot 3 + 20 = 20 + 18 - 9 = 29$ft.

6.(6 pts.) The graph of the piecewise defined function $g(x)$ is shown below. The graph consists of part of a circle and straight lines. Use the graph to calculate $\int_0^8 g(x)\, dx$.



Solution:

We need to compute the area between the function and the x-axis. Areas below the x-axis are counted as negative, and areas above the x-axis are counted as positive. From $x = 0$ to $x = 4$, we have a quarter of a circle of radius 4 that lies below the x-axis, so

$$\int_0^4 g(x)dx = -\frac{1}{4}\left(\pi(4)^2\right) = -4\pi.$$

From $x = 4$ to $x = 7$, we have a triangle with base 3 and height 6 that lies above the x-axis, so

$$\int_4^7 g(x)dx = \frac{1}{2}(3)(6) = 9.$$

From $x = 7$ to $x = 8$, we have a triangle that lies below the x-axis with base 1 and height 6, so

$$\int_7^8 g(x)dx = -\frac{1}{2}(1)(6) = -3.$$

Thus,

$$\int_0^8 g(x)dx = \int_0^4 g(x)dx + \int_4^7 g(x)dx + \int_7^8 g(x)dx = -4\pi + 9 - 3 = 6 - 4\pi.$$

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7.(6 pts.) Let

$$h(x) = \int_1^{x^2} \frac{1}{4+\sin^2(t)}\, dt.$$

Find $h'(x)$.

Solution:

Let $f(x) = \int_1^x \frac{1}{4+\sin^2 t} dt$, and let $g(x) = x^2$. Then $h(x) = (f \circ g)(x)$. Therefore, by the chain rule, $h'(x) = f'(g(x))g'(x)$. Since $f'(x) = \frac{1}{4+\sin^2 x}$ by the Fundamental Theorem of Calculus,

$$h'(x) = \frac{1}{4+\sin^2(x^2)}(2x) = \frac{2x}{4+\sin^2(x^2)}$$

8.(6 pts.) Calculate the indefinite integral

$$\int \frac{x+\sin(\sqrt{x})}{\sqrt{x}}; dx.$$

Solution:

First, rewrite $\int \frac{x+\sin(\sqrt{x})}{\sqrt{x}}\, dx = \int x^{1/2}\, dx + \int \frac{\sin(x^{1/2})}{x^{1/2}}\, dx$. We have

$$\int x^{1/2}\, dx = \frac{2}{3}x^{3/2} + C.$$

To evaluate the other integral, we need to use substitution. Let $u = x^{1/2}$, then $du = \frac{1}{2}x^{-1/2}\, dx \quad \Leftrightarrow \quad dx = 2x^{1/2}du$. Then,

$$\int \frac{\sin(x^{1/2})}{x^{1/2}}\, dx = \int \frac{\sin(x^{1/2})}{x^{1/2}}\, (2x^{1/2}du) = \int 2\sin u\, du = -2\cos(u)+C = -2\cos(\sqrt{x})+C.$$

Therefore, $\int \frac{x+\sin(\sqrt{x})}{\sqrt{x}}\, dx = \frac{2}{3}x^{3/2} - 2\cos(\sqrt{x}) + C.$

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9.(6 pts.) An underground beer pipeline in the city of Bruges has sprung a leak which is gradually worsening. Your statistics suggest that beer is leaking from the pipeline at a rate of $3t^2 - 4t + 3$ gallons per day, where t denotes the number of days after the leak started. How many gallons of beer will have leaked in the first 2 days after the leak started?

Solution:
The amount of beer leaked in the first two days is given by

$$\int_0^2 3t^2 - 4t + 3dt$$

gallons. Solving,

$$\begin{aligned}\int_0^2 3t^2 - 4t + 3dt &= \left(t^3 - 2t^2 + 3t\right)\Big|_0^2 \\ &= 2^3 - 2 \cdot 2^2 + 3 \cdot 2 - 0 \\ &= 8 - 8 + 6 \\ &= 6 \text{ gallons}\end{aligned}$$

10.(6 pts.) Evaluate the following definite integral

$$\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\, dx$$

Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x} dx &= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} \frac{1}{\cos^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \tan x \sec^2 x dx\end{aligned}$$

Now we can use u-substitution. Let $u = \tan x$. Then $du = \sec^2 x dx$, so

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\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x} dx &= \int_0^{\frac{\pi}{4}} \tan x \sec^2 x dx \\
&= \int_{\tan(0)}^{\tan\left(\frac{\pi}{4}\right)} u du \\
&= \int_0^1 u du \\
&= \frac{1}{2} u^2 |_0^1 \\
&= \frac{1}{2}.
\end{aligned}
$$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(13 pts.) (a) Evaluate the definite integral $\int_0^2 x^3\,dx$ using the right endpoint approximation and the **limit definition** of the definite integral.

Hint: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Solution:

By the limit definition of the integral, we have

$$\begin{aligned}
\int_0^2 x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{16i^3}{n^4} \\
&= \lim_{n\to\infty} \frac{16}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{16}{n^4} \left(\frac{n(n+1)}{2}\right)^2 \\
&= \lim_{n\to\infty} \frac{4(n^2+2n+1)}{n^2} \\
&= \lim_{n\to\infty} \frac{4 + \frac{8}{n} + \frac{4}{n^2}}{1} \\
&= 4.
\end{aligned}$$

(b) Verify your answer using the fundamental theorem of calculus.

We have

$$\int_0^2 x^3 dx = \frac{1}{4}x^4\big|_0^2 = \frac{1}{4}(2^4) - 0 = 4.$$

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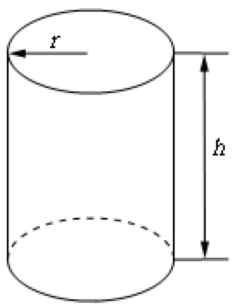
12.(13 pts.) A manufacturer needs to make a cylindrical can (top included) that will hold 2000 cm^3 of liquid. Find the dimensions of the can (values of r and h) that will minimize the amount of material used to make the can.
(Exact values such as $\sqrt{2}, \sqrt{3}, \pi, \sqrt{\pi}, etc$ ... should not be converted to a decimal approximation.)



Note that the surface area of a cylinder with no top or bottom is $2\pi rh$ cm^2.

Solution:
To minimize the amount of material used to make the can, we need to minimize its surface area, including the top and bottom. The surface area function of the cylinder with top and bottom is given by

$$S = 2\pi r^2 + 2\pi rh.$$

Now, in order to take the derivative of S, we need to rewrite S in term of one and only one variable. To accomplish this, we are given the fact that the volume of this can needs to be 2000 cm^3. So, $V = \pi r^2 h = 2000 \quad \Rightarrow \quad h = \dfrac{2000}{\pi r^2}$. Then,

$$S = 2\pi r^2 + 2\pi r \left(\frac{2000}{\pi r^2}\right) = 2\pi r^2 + \frac{4000}{r}.$$

So,

$$S' = 4\pi r - \frac{4000}{r^2} = 0 \Leftrightarrow 4\pi r^3 = 4000 \Leftrightarrow r^3 = \frac{4000}{4\pi} \Leftrightarrow r = \frac{10}{\sqrt[3]{\pi}}$$

Note that we don't consider the critical point $r = 0$ in this case since we want $r > 0$. And $S'' = 4\pi + \dfrac{8000}{r^3} > 0$ for all $r > 0$. Thus, at the critical point $r = \dfrac{10}{\sqrt[3]{\pi}}$, the surface area function attains its minimal value. So then

$$h = \frac{2000}{\pi r^2} = \frac{2000}{\pi} \cdot \left(\frac{\sqrt[3]{\pi}}{10}\right)^2 = \frac{2000}{100} \cdot \frac{\pi^{2/3}}{\pi} = \frac{20}{\sqrt[3]{\pi}}.$$

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Answer: $r = \dfrac{10}{\sqrt[3]{\pi}}$ and $h = \dfrac{20}{\sqrt[3]{\pi}}$.

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13.(12 pts.) Find the area of the bounded region between the curves

$$y = x^2 - 2x + 1 \quad \text{and} \quad y = 7 - x^2 + 2x.$$

Solution:

Let

$$\begin{aligned} f(x) &= (7 - x^2 + 2x) - (x^2 - 2x + 1) \\ &= -2x^2 + 4x + 6 \\ &= -2(x^2 - 2x - 3) \\ &= -2(x + 1)(x - 3) \end{aligned}$$

be the difference between these two curves. The area between the curves is the area under $f(x)$ between that function's x-intercepts. We choose the difference that we did, rather than subtracting in the opposite order, because it makes the leading coefficient negative. Consequently, the region of the resulting f between the x-intercepts is above the x-axis.

Since the above factorization yields x-intercepts of -1 and 3, the area of interest is given by

$$\begin{aligned} \int_{-1}^{3} -2x^2 + 4x + 6dx &= \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Big|_{-1}^{3} \\ &= \left(-\frac{2}{3}3^3 + 2 \cdot 3^2 + 6 \cdot 3\right) \\ &\quad - \left(-\frac{2}{3}(-1)^3 + 2(-1)^2 + 6(-1)\right) \\ &= (-18 + 18 + 18) - \left(\frac{2}{3} + 2 - 6\right) \\ &= 18 - \frac{2}{3} + 4 \\ &= 21 + \frac{1}{3} \\ &= \frac{64}{3} \end{aligned}$$

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14.(2 pts.) You will earn 2 points if your instructor can read your name easily on the front page of the exam and you mark the answer boxes with an X (as opposed to a circle or any other mark).

Rough Work