Name: ______________________________

Instructor: __________________________

.

Multiple Choice

1.(6 pts.) A cylinder has constant height $h = 2$ m, but the radius is changing. If the volume is increasing at a rate of 16 m^3/sec., how fast is the radius changing when the radius is 4 m.

Solution:
The formula for the volume of a cylinder is $V = \pi r^2 h$. Since $h = 2$ is constant, we can write $V = 2\pi r^2$. Now, differentiating both sides of the equation with respect to t, we get

$$\frac{dV}{dt} = 4\pi r \frac{dr}{dt}$$

We know $\dfrac{dV}{dt} = 16$; and the problem asks for $\dfrac{dr}{dt}$ when $r = 4$. Plugging these values into the above equation, we find that

$$16 = 4\pi(4)\frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{1}{\pi}$$

2.(6 pts.) A beetle is moving along a straight line, with position given by $s(t) = \sin(t) + \cos(t)$. How much distance does it travel from $t = 0$ to $t = \pi/3$?

Solution:
We need to find the points where the beetle might change direction, and look at its position at those times. The direction of travel is governed by the sign of the velocity, i.e. the sign of $s'(t)$. Calculate $s'(t)$:

$$s'(t) = \cos(t) - \sin(t).$$

Since $s'(t)$ is continuous, $s'(t)$ can only change sign by crossing through zero. Therefore, we need to find the points where $s'(t) = 0$, i.e., the points where $\sin(t) = \cos(t)$. Thinking geometrically, this occurs on a right triangle exactly when the lengths of the legs (non-hypotenuse sides) are equal. In the interval $\left(0, \dfrac{\pi}{3}\right)$, we have this equality exactly at an angle of $\dfrac{\pi}{4}$.

Name: ___________________________

Instructor: ________________________

We therefore compute the total distance travelled as follows:

$$\begin{aligned}
\text{total distance} &= \left|s\left(\frac{\pi}{4}\right) - s(0)\right| + \left|s\left(\frac{\pi}{3}\right) - s\left(\frac{\pi}{4}\right)\right| \\
\left|s\left(\frac{\pi}{4}\right) - s(0)\right| &= \left|\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \sin(0) - \cos(0)\right| \\
&= \left|\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right| \\
&= \sqrt{2} - 1 \\
\left|s\left(\frac{\pi}{3}\right) - s\left(\frac{\pi}{4}\right)\right| &= \left|\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right| \\
&= \left|\frac{\sqrt{3}}{2} + \frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right| \\
&= \sqrt{2} - \frac{1+\sqrt{3}}{2} \\
\text{total distance} &= \sqrt{2} - 1 + \sqrt{2} - \frac{1+\sqrt{3}}{2} \\
&= 2\sqrt{2} - \frac{3}{2} - \frac{\sqrt{3}}{2}
\end{aligned}$$

Name: ______________________________

Instructor: __________________________

3.(6 pts.) Find the linearization $L(x)$ of the function $f(x) = \tan(x)$ at $\frac{\pi}{4}$.

Solution:

We know the linearization of f at a point a is

$$L(x) = f(a) + f'(a)(x - a).$$

Here, we have $a = \frac{\pi}{4}$, so we compute

$$f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1,$$

and since $f'(x) = \sec^2(x)$,

$$f'(a) = f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = 2.$$

Thus,

$$L(x) = 1 + 2\left(x - \frac{\pi}{4}\right) = 1 - \frac{\pi}{2} + 2x.$$

4.(6 pts.) Use linear approximation of $f(x) = \sqrt{3+x}$ at $a = 1$ to estimate $\sqrt{3.6}$.

Solution:

We will use the formula $L(x) = f(a) + f'(a)(x - a)$, where $a = 1$. Our estimate for $\sqrt{3.6}$ will then be $L(.6)$. First, note that $f(1) = 2$, and

$$f'(x) = \frac{1}{2\sqrt{3+x}}$$

In particular, $f'(a) = f'(1) = \frac{1}{4}$. Putting this all together, we have that:

$$L(x) = 2 + \frac{1}{4}(x - 1).$$

Finally, letting $x = .6$, $L(.6) = 1.9$.

Name: ____________________________

Instructor: ________________________

5.(6 pts.) Consider the function $f(x) = x^{1/3}(x+1)^2$. Which of the following is a complete list of the critical points of f?

Solution:

First, we differentiate f,

$$\begin{aligned}
f'(x) &= x^{1/3}\frac{d}{dx}\left[(x+1)^2\right] + \left(\frac{d}{dx}\left[x^{1/3}\right]\right)(x+1)^2 \\
&= x^{1/3}(2)(x+1) + \frac{1}{3}x^{-2/3}(x+1)^2 \\
&= x^{-2/3}(x+1)\left[2x + \frac{1}{3}(x+1)\right] \\
&= x^{-2/3}(x+1)\left[\frac{7}{3}x + \frac{1}{3}\right] \\
&= \frac{1}{3}x^{-2/3}(x+1)(7x+1) \\
&= \frac{(x+1)(7x+1)}{3x^{2/3}}
\end{aligned}$$

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. The domain of f is $(-\infty, \infty)$. Now

$$\begin{aligned}
f'(x) = \frac{(x+1)(7x+1)}{3x^{2/3}} &= 0 \\
\implies (x+1)(7x+1) &= 0 \\
&\implies x = -1 \text{ or } x = -\frac{1}{7}
\end{aligned}$$

and $f'(x)$ is undefined when $x = 0$. Thus the critical numbers of $f(x)$ are given by $x = 0, -\frac{1}{7}, -1$.

6.(6 pts.) Let

$$f(\theta) = \frac{\theta^2}{4} + \cos(\theta) \quad \text{where} \quad 0 \le \theta \le \pi.$$

Which of the following statements is true about the graph of f?

Solution:

Name: ______________________________

Instructor: ____________________________

The concavity is controlled by the sign of the second derivative. Calculate:

$$f'(\theta) = \frac{\theta}{2} - \sin(\theta)$$
$$f''(\theta) = \frac{1}{2} - \cos(\theta)$$

We can see that f'' is negative (and f is concave down) exactly when $\cos(\theta) > \frac{1}{2}$. This, in turn, occurs exactly when $\theta < \frac{\pi}{3}$. The function is therefore concave down on $(0, \frac{\pi}{3})$, and concave up on $(\frac{\pi}{3}, \pi)$.

Name: ____________________________

Instructor: ________________________

7.(6 pts.) Consider the function $f(x) = x^3 - 3x^2 - 9x + 2014$. Which of the following statements is true?

Solution:
We need to investigate the local extrema and points of inflection of f. First, we find

$$f'(x) = 3x^2 - 6x - 9.$$

The critical points are the points where $f'(x) = 0$ or $f'(x)$ is undefined. If $f'(x) = 0$, then

$$3x^2 - 6x - 9 = 0 \Longrightarrow (x+1)(3x-9) = 0,$$

so we have critical points when $x = -1$ and when $x = 3$. Since $f'(x)$ is a polynomial and is therefore defined everywhere, these are our only critical points. For $x < -1$, $f'(x) > 0$, for $-1 < x < 3$, $f'(x) < 0$, and for $x > 3$, $f'(x) > 0$. Thus, f has a local maximum at $x = -1$ and f has a local minimum at $x = 3$.

Points of inflection occur when $f''(x) = 0$. We have

$$f''(x) = 6x - 6.$$

Then $f''(x) = 0$ when $6x - 6 = 0$, so we have a point of inflection at $x = 1$.

8.(6 pts.) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{x-4}$.

Solution:
Multiplying the numerator and denominator by $\dfrac{1}{x}$ gives:

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{x-4} = \lim_{x\to-\infty} \frac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(x-4)} = \lim_{x\to-\infty} -\frac{\sqrt{2+\frac{1}{x^2}}}{1-\frac{4}{x}} = -\sqrt{2}.$$

The sign in third expression comes from the fact that when $x < 0$, $\dfrac{1}{x} = -\sqrt{\dfrac{1}{x^2}}$.

Name: ______________________________

Instructor: __________________________

9.(6 pts.) The **derivative and second derivative** of the function $f(x)$ are given by

$$f'(x) = \frac{(x-2)(x-3)}{x} \qquad \text{and} \qquad f''(x) = 1 - \frac{6}{x^2} \; .$$

On which of the following intervals is $f(x)$ it both decreasing and concave up?

Solution:

Since $f'(x) = \dfrac{(x-2)(x-3)}{x}$, the critical numbers of f are $x = 0, 2, 3$. We analyze f in the following table

interval	sign of $f'(x) = \dfrac{(x-2)(x-3)}{x}$	f
$x < 0$	$-$	decreasing on $x < 0$
$0 < x < 2$	$+$	increasing on $0 < x < 2$
$2 < x < 3$	$-$	decreasing on $2 < x < 3$
$3 < x$	$+$	increasing on $3 < x$

For concavity, $f''(x) = 1 - \dfrac{6}{x^2} = 0 \;\Rightarrow\; x = \pm\sqrt{6}$. We have the following table about the concavity of f:

interval	sign of $f''(x) = 1 - \dfrac{6}{x^2}$	f
$x < -\sqrt{6}$	$+$	concave up on $x < -\sqrt{6}$
$-\sqrt{6} < x < \sqrt{6}$	$-$	concave down on $-\sqrt{6} < x < \sqrt{6}$
$\sqrt{6} < x$	$+$	concave up on $\sqrt{6} < x$

We know $-\sqrt{6} < 0 < \sqrt{4} = 2 < \sqrt{6} < 3 = \sqrt{9}$. So, according to our tables, f is both decreasing and concave up on $x < -\sqrt{6}$ or $\sqrt{6} < x < 3$.

10.(6 pts.) What is the minimum value of the function $f(t) = 2t^3 - 3t^2 - 12t + 6$ on the interval $[-2, 3]$?

Solution:

Since f is continuous and $[-2, 3]$ is a closed interval, the Extreme Value Theorem tells us that this function attains an absolute minimum value on this interval. Fermat's theorem then tells us that this minimum value must occur at a critical point or at an endpoint. We proceed by finding the critical points, and identifying the least value f attains on the mentioned points.

Calculate:

$$f'(t) = 6t^2 - 6t - 12.$$

Name: ______________________________

Instructor: __________________________

We know $f'(t)$ is defined everywhere, so the critical points occur exactly when $f'(t) = 0$. Factor $f'(t)$ as:

$$f'(t) = 6(t^2 - t - 2) = 6(t-2)(t+1).$$

The points we must check are therefore -2, -1, 2, and 3. Observe:

$$\begin{aligned} f(-2) &= -2 * 8 - 3 * 4 + 24 + 6 = 2 \\ f(-1) &= -2 - 3 + 12 + 6 = 13 \\ f(2) &= 2 * 8 - 3 * 4 - 12 * 2 + 6 = -14 \\ f(3) &= 2 * 27 - 3 * 9 - 12 * 3 + 6 = -3 \end{aligned}$$

The minimum value is therefore -14, which is attained at $x = 2$.

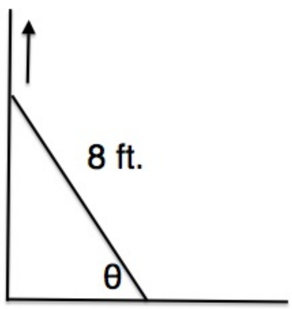
Name: ______________________________

Instructor: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(12 pts.) A ladder 8 ft long leans against a vertical wall. The top of the ladder is pulled up from the floor at a rate of 2 ft/second. Let θ be the angle between the ladder and the ground. Find $\dfrac{d\theta}{dt}$ when the bottom of the ladder is 4 ft away from the wall.



Solution:
Let y be the distance from the bottom of the wall to the top of the ladder. Then $\sin(\theta) = \dfrac{y}{8}$. Implicit differentiation with respect to t gives us

$$\cos(\theta)\frac{d\theta}{dt} = \frac{1}{8}\frac{dy}{dt}$$

Let x be the distance from the bottom of the wall to the bottom of the ladder. Then $\cos(\theta) = \dfrac{x}{8}$. When the bottom of the ladder is 4 feet from the wall, we have $x = 4$, so

$$\cos(\theta) = \frac{4}{8} = \frac{1}{2}.$$

Since the ladder is pulled up from the floor at a rate of 2 ft/second, y increases at a rate of 2 ft/second, so $\dfrac{dy}{dt} = 2$. Plugging these values into our equation above gives

$$\frac{1}{2}\frac{d\theta}{dt} = \frac{1}{8}(2) \Longrightarrow \frac{d\theta}{dt} = \frac{1}{2}.$$

Name: ____________________________

Instructor: _________________________

12.(12 pts.) Show that the equation

$$x^7 + 2x^5 + 5x + 4 = 0$$

has one and exactly one real solution. Identify the theorem(s) you are using.

Solution:
Let $p(x) = x^7 + 2x^5 + 5x + 4$. First, we will apply the Intermediate Value Theorem to argue for the existence of a root. This isn't so difficult, as $p(0) = 4$ and $p(-1) = -4$. By the IVT (which we can apply because $p(x)$ is continuous), this sign change guarantees a root in the interval $(-1, 0)$. I will call this root a. Why is this root unique? To answer this question, let's examine this polynomial's derivative:

$$p'(x) = 7x^6 + 10x^4 + 5$$

One can easily see that this derivative is always positive (as each term will always be nonnegative). This means the function is always strictly increasing, i.e. $p(x) > p(a)$ if $x > a$ and $p(x) < p(a)$ if $x < a$. In particular, $p(x) \neq 0$ if $x \neq a$. In other words, a is the unique root of this polynomial.

We can also use Rolle's theorem to see that we have only one root. If a and b are two distinct roots (let's say $a < b$), then $f(a) = 0 = f(b)$, and since $p(x)$ is continuous and differentiable everywhere, Rolle's theorem says there must exist some c such that $a < c < b$ and $f'(c) = 0$. But, this is of course nonsense, as we have already argued that f' is always positive. Therefore, there must be only one root. Well isn't that nice? Calculus solves a problem in algebra.

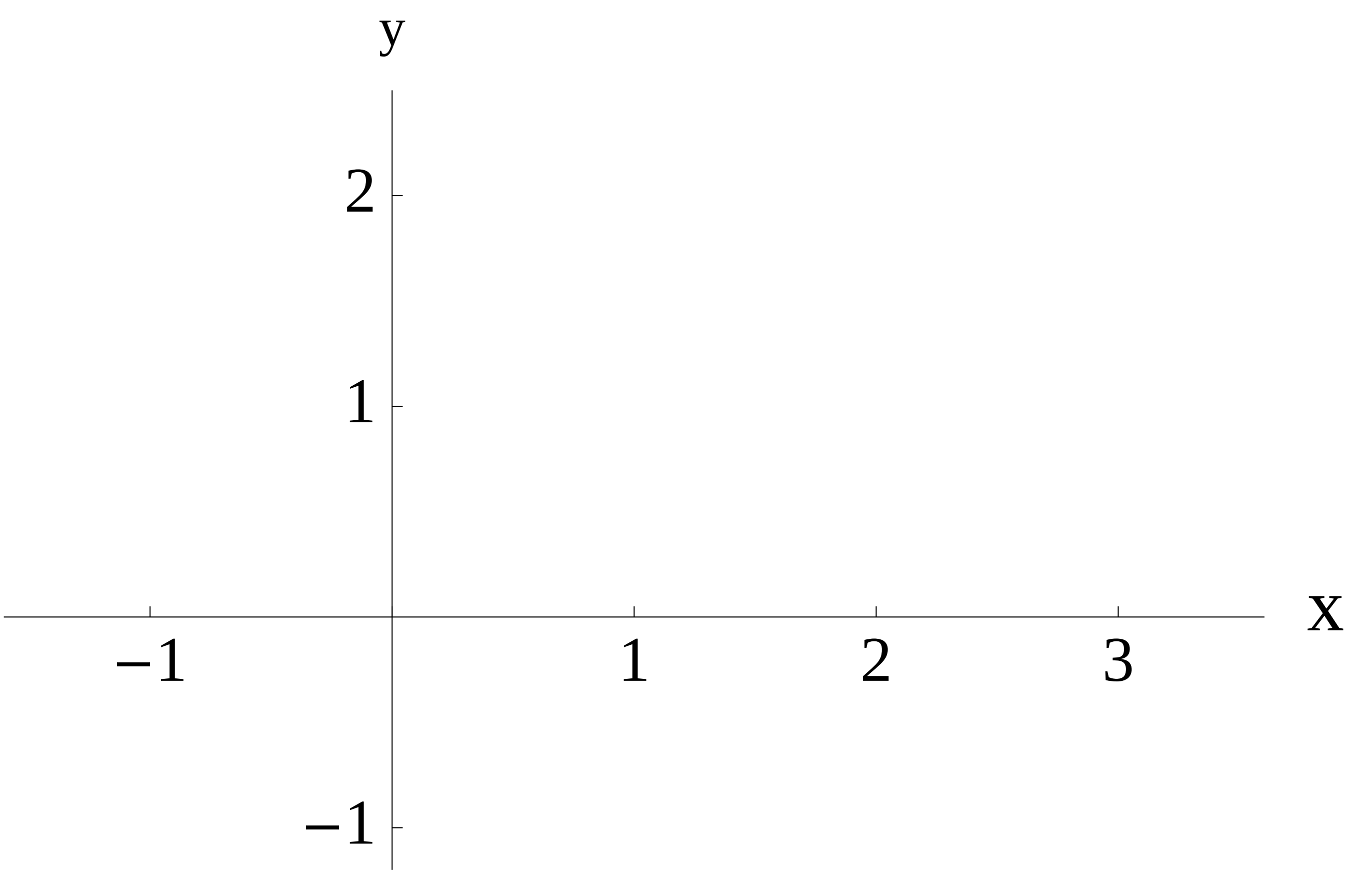
Name: ______________________

Instructor: ______________________

13.(13 pts.)
The table below shows what is known about a function f which is **defined and continuous** on the interval $[-1, 3]$. The table gives the values (or the sign) of f, f' and f'' at the points given (D.N.E indicates that the derivative does not exist at that point) and tells whether f' and f'' are positive or negative on the intervals given.

x	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, 3)$	3
$f(x)$	2		1		0		1		-0.5
$f'(x)$		< 0		< 0	0	> 0	$D.N.E.$	< 0	
$f''(x)$		< 0		> 0	> 0	> 0		> 0	

Sketch the graph of a function $f(x)$ satisfying the above data.

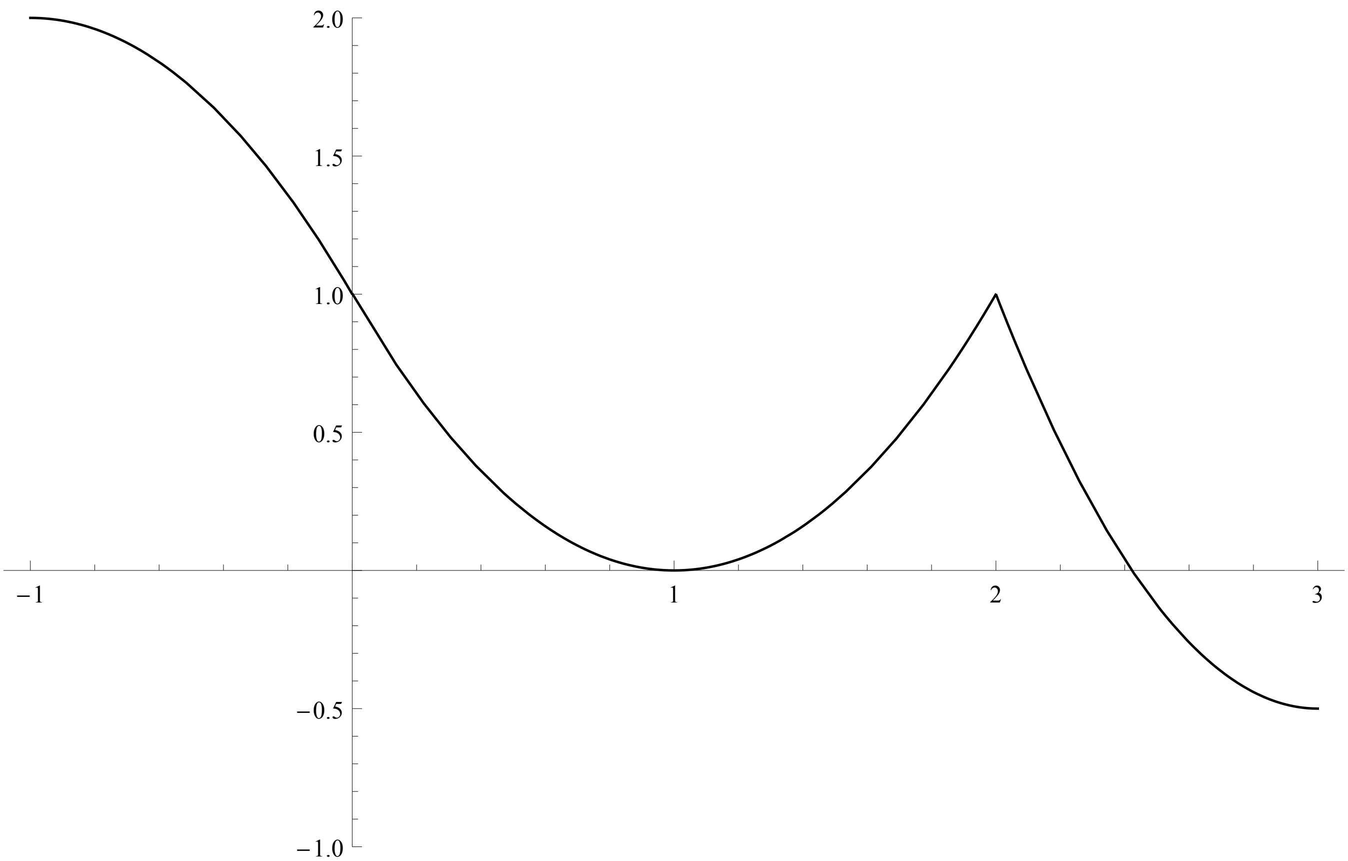


Name: ______________________________

Instructor: __________________________

Solution: First, we want to make sure that our graph goes through these points: $(-1,2),(0,1),(1,0),(2,1),(3,-0.5)$. According to the table, f decreases on the interval $(-1,1)$, increases on $(1,2)$, but decreases again on $(2,3)$. Note also that we have a horizontal tangent line at $x=1$ since $f'(1)=0$ and a sharp turn at $x=2$ because $f'(2)$ is undefined but f is defined and continuous there. Next, we want to make sure that our graph has correct concavity. The function f is concave down on $(-1,0)$, upward from $x=0$ all the way to $x=3$. Note that the inflection point is at $x=0$ since f changes concavity there.

Below is an example of the graph of f:



Name: ____________________________

Instructor: ________________________

14.(3 pts.) You will earn 3 points if your instructor can read your name easily on the front page of the exam and you mark the answer boxes with an X (as opposed to a circle or any other mark).