Name: ______________________________

Instructor: __________________________

Math 10550, Exam I
November 10, 2013

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
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7.	(a)	(b)	(c)	(d)	(e)
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Multiple Choice	________
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Multiple Choice

1.(7 pts.) Starting at time $t = 0$ a particle is oscillating vertically. After t minutes the height of the particle above ground (*in feet*, upward is positive) is given by

$$10\cos(\pi t).$$

Which one of the statements below is correct when $t = 0.25$ minutes? (*Only one is*)

Solution: Let $h(t) = 10\cos(\pi t)$.

$$h'(t) = -10\pi\sin(\pi t), \qquad h''(t) = -10\pi^2\cos(\pi t).$$

$$h(\frac{1}{4}) = 10\cos(\frac{\pi}{4}) = \frac{10}{\sqrt{2}} > 0.$$

$$h'(\frac{1}{4}) = -10\pi\sin(\frac{\pi}{4}) = -\frac{10\pi}{\sqrt{2}} < 0.$$

$$h''(\frac{1}{4}) = -10\pi^2\cos(\frac{\pi}{4}) = -\frac{10\pi^2}{\sqrt{2}} < 0.$$

Since $h(0.25) > 0$, the particle is above ground and since $h'(0.25) < 0$, the particle is descending. Because $h'(0.25)$ and $h''(0.25)$ have the same sign, the particle is speeding up when $t = 0.25$.

(a) The particle is below ground, ascending and slowing down.

(b) The particle is below ground, descending and speeding up.

(c) The particle is above ground, descending and speeding up.

(d) The particle is above ground, descending and slowing down.

(e) The particle is above ground, ascending and slowing down.

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2.(7 pts.) Let f be a function which is continuous on the interval $[0, 18]$ and differentiable on $(0, 18)$. If $f(0) = 1$ and

$$|f'(x)| \leq 2 \quad \text{for all} \quad x \in (0, 18),$$

which statement below **must** be true? (*only one must be*, the remaining ones ***might*** be false)

Solution: Since f is continuous on $[0, 4]$ and differentiable on $(0, 4)$, the Mean Value Theorem applies on this interval and $\dfrac{f(4) - f(0)}{4 - 0} = f'(c)$ for some number c in $(0, 4)$. Therefore

$$-2 \leq \frac{f(4) - f(0)}{4} \leq 2$$

and

$$-8 \leq f(4) - 1 \leq 8$$

which gives

$$-7 \leq f(4) \leq 9.$$

(a) $|f(4)| \leq 2$

(b) $f(x) = 1 + 2x$

(c) $-1 \leq f(4) \leq 3$

(d) $-7 \leq f(4) \leq 9$

(e) $f'(4) = 2$

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3.(7 pts.) If $f'(x) = \dfrac{(x-1)^2 x}{(x+1)^3}$, find the local maxima and minima of $f(x)$ assuming that the domain of $f(x)$ is all $x \neq -1$. (Note: you are given f', not f.)

Solution: It is clear that for the numerator to be 0, x must be 0 or 1. We use the first derivative test to determine if either of these is a local maximum or minimum. If $-1 < x < 0$, the numerator is negative and the denominator is positive, making f' negative. If $x > 0$ the numerator is positive and the denominator is positive, making f' positive. Hence 0 is a local minimum. We also see 1 is neither a local minimum or maximum because on either side of 1, f' is positive.

(a) f has a local minimum at $x = 0$; there is no local maximum

(b) f has a local minimum at $x = 1$; f has a local maximum at $x = -1$

(c) f has a local minimum at $x = 0$; f has local maxima at $x = 1$ and $x = -1$

(d) f has a local minimum at $x = 0$; f has a local maximum at $x = 1$

(e) There are no local minima or local maxima

4.(7 pts.) . Evaluate $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^6+5}}{x^3+1}$.

Solution: We first divide through by x^3, in the numerator using the fact that $\sqrt{x^6} = -x^3$ for $x < 0$:

$$\frac{\sqrt{4x^6+5}}{x^3+1} = -\frac{\sqrt{4+5/(x^6)}}{1+1/(x^3)}.$$

As x goes to $-\infty$, the numerator goes to $\sqrt{4} = 2$. As x goes to $-\infty$, the denominator goes to 1. Hence the answer is -2.

(a) 3/2 (b) -2 (c) 2 (d) 6 (e) 4

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5.(7 pts.) How many inflection points does the curve $y = 4x^5 - 5x^4 - 12$ have?

Solution: Set $f(x) = 4x^5 - 5x^4 - 12$. If x is an inflection point for f, we must have $f''(x) = 0$.

$$f''(x) = 80x^3 - 60x^2 = 20x^2(4x - 3).$$

Thus our possibilities for inflection points are $x = 0$ and $x = 3/4$. If $x < 0$ then $f''(x)$ is negative. If $0 < x < 3/4$ then $f''(x)$ remains negative. Hence $f(x)$ remains concave down before and after $x = 0$, so it is not an inflection point. If $x > 3/4$ then $f''(x)$ is positive. Hence the concavity of f changes from downward to upward at $x = 3/4$, so that is the one and only inflection point.

(a) 2 (b) 3 (c) None (d) 4 (e) 1

6.(7 pts.) Suppose $f(x)$ is continuous and differentiable for all real numbers.
If $-1 \leq f'(x) \leq 3$ and $f(5) = 6$, what is the largest $f(x)$ can be at $x = 1$?

Solution: By the Mean Value Theorem, there exists a c in $[1, 5]$ such that

$$\frac{f(5) - f(1)}{5 - 1} = f'(c),$$

or

$$f(1) = f(5) - 4f'(c) = 6 - 4f'(c).$$

To find the largest $f(1)$, we need to take $f'(c)$ to be the smallest, $f'(c) = -1$. Hence the largest $f(1)$ is $6 - 4(-1) = 10$.

(a) -6 (b) 10 (c) 2 (d) 11 (e) 18

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7.(7 pts.) Find the linearization $L(x)$ of the function $f(x) = (3x+125)^{1/3}$ at $a = 0$

Solution: First, $f(0) = 125^{1/3} = 5$. Next,

$$f'(x) = \frac{3}{3(3x+125)^{2/3}} = \frac{1}{(3x+125)^{2/3}}.$$

Hence $f'(0) = \dfrac{1}{125^{2/3}} = \dfrac{1}{25}$. Therefore,

$$L(x) = \frac{1}{25}x + 5.$$

(a) $\frac{1}{5}x + 5$

(b) $\frac{3}{25}x + \frac{1}{25}$

(c) $\frac{1}{25}x - 5$

(d) $\frac{3}{25}(x-1) + 5$

(e) $\frac{1}{25}x + 5$

8.(7 pts.) Use the linear approximation (or tangent line approximation) of $f(x) = \cos(x)$ at $x = \pi/2$ to find approximate value of $f(x)$ at $x = 3\pi/5$.

Solution: The derivative of $f(x) = \cos(x)$ is

$$f'(x) = -\sin(x)$$

and so we have $f(\pi/2) = 0$ and $f'(\pi/2) = -1$. Putting these values into the linear approximation, we see that the linearization is

$$f(x) \approx f(\pi/2) + f'(\pi/2)(x - \pi/2) = -(x - \pi/2).$$

So

$$f(3\pi/5) \approx -(3\pi/5 - \pi/2) = -\pi/10.$$

(a) $\frac{\pi}{5}$ (b) $-\frac{1}{10}$ (c) $\frac{\pi}{10}$ (d) $-\frac{\pi}{10}$. (e) $-\frac{1}{7}$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

9.(11 pts.) First answer the 8 questions below. Then use your answers to graph $y = x + 2\cos x$ on the interval $[0, 2\pi]$. ($\pi \approx 3.14$, $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$)

1a) $y' =$

1b) On what interval(s) is y decreasing?

1c) Give both coordinates of any local maxima.

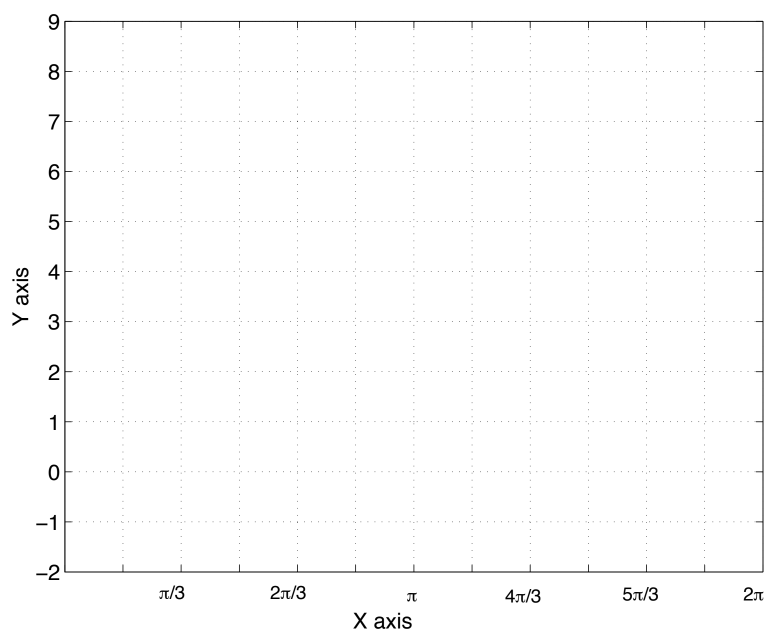
1d) Give both coordinates of any local minima.

2a) $y'' =$

2b) On what interval(s) is y concave down?

2c) Give both coordinates of any points of inflection.

2d) Give the slope of the tangent line at any points of inflection.



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Solution: 1a) $y' = 1 - 2\sin x$.
1b) Note

$$0 > 1 - 2\sin x \;\Leftrightarrow\; 2\sin x > 1 \Leftrightarrow \sin x > \frac{1}{2} \;\Leftrightarrow\; \frac{\pi}{6} < x < \frac{5\pi}{6}.$$

So on the interval $[\frac{\pi}{6}, \frac{5\pi}{6}]$, y is decreasing.
1c)-1d) Note

$$1 - 2\sin x = 0 \;\Leftrightarrow\; 2\sin x = 1 \;\Leftrightarrow\; \sin x = \frac{1}{2} \;\Leftrightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Since y is decreasing on $[\frac{\pi}{6}, \frac{5\pi}{6}]$ and increasing elsewhere, $f(\frac{\pi}{6}) = \frac{\pi}{6} + \sqrt{3}$ is a local maxima and $f(\frac{5\pi}{6}) = \frac{5\pi}{6} - \sqrt{3}$ is a local minima.
2a) $y'' = -2\cos(x)$.
2b) Note

$$-2\cos x < 0 \;\Leftrightarrow\; \cos x > 0 \;\Leftrightarrow\; 0 < x < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < x < 2\pi.$$

So on the interval $[0, \frac{\pi}{2}]$ and $[\frac{3\pi}{2}, 2\pi]$, y is concave down.
2c) From 2b), we can see that $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, \frac{3\pi}{2})$ are infection points.
2d) At $(\frac{\pi}{2}, \frac{\pi}{2})$, the slope of the tangent line is $1 - 2\sin(\pi/2) = -1$.
At $(\frac{3\pi}{2}, \frac{3\pi}{2})$, the slope of the tangent line is $1 - 2\sin(3\pi/2) = 3$.

Second Solution

1a) $y' = 1 - 2\sin x$

1b) On what interval(s) is y decreasing? y is decreasing when $y' < 0$, i.e. when $1-2\sin x < 0$ or $\sin x > 1/2$. This happens when $\frac{\pi}{6} < x < \frac{5\pi}{6}$. ($y'$ is positive elsewhere on the interval $[0, 2\pi]$.)

1c) Give both coordinates of any local maxima. y' switches sign from $+$ to $-$ at $x = \frac{\pi}{6}$, hence we have a local max. at $x = \frac{\pi}{6}$ with coordinates $(\frac{\pi}{6}, f(\frac{\pi}{6})) = (\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}) \approx (\frac{\pi}{6}, .52 + 1.73 = 2.25)$, where $f(x) = 1 - 2\sin x$

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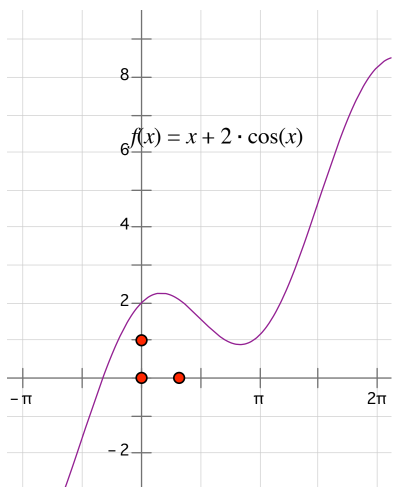
1d) Give both coordinates of any local minima. y' switches sign from $-$ to $+$ at $x = \frac{5\pi}{6}$, hence we have a local min. at $x = \frac{5\pi}{6}$ with coordinates $(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}) \approx (\frac{5\pi}{6}, 2.62 - 1.73 = .89)$.

2a) $y'' = \ -2\cos x$

2b) On what interval(s) is y concave down? $y'' = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$. y is concave down when $y'' < 0$ i.e. when $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$. Draw a sketch to see this.

2c) Give both coordinates of any points of inflection. You will see from your sketch of the sign of y'' that y'' switches sign at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Hence both give points of inflection on the graph at $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, \frac{3\pi}{2})$.

2d) Give the slope of the tangent line at any points of The slope of the tangent to the graph at the points of inflection are given by the value of the derivative at the point. Slope of tangent at $x = \frac{\pi}{2} = 1 - 2\sin(\frac{\pi}{2}) = 1 - 2 = -1$. Slope of tangent at $x = \frac{3\pi}{2} = 1 - 2\sin(\frac{3\pi}{2}) = 1 + 2 = 3$. inflection.



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10.(10 pts.) Find the extreme values of $f(x) = 3|x| - x^2 - 2$ on $-1 \leq x \leq 2$.

Solution: When $-1 \leq x \leq 0$, $f(x) = -3x - x^2 - 2$ and hence $f'(x) = -3 - 2x$. So the critical number of f is $x = -3/2$. However $x = -3/2$ is not in the domain.

When $0 \leq x \leq 2$, $f(x) = 3x - x^2 - 2$ and hence $f'(x) = 3 - 2x$. So the critical number of f is $x = 3/2$.

Therefore, the critical numbers of f are $x = 0, 3/2$. We compute $f(-1) = 0$, $f(0) = -2$, $f(3/2) = 1/4$ and $f(2) = 0$. So the extreme values are -2 and $1/4$.

Second Solution

This is a continuous function on the closed interval $[-1, 2]$, so by the extreme value theorem it must have and absolute maximum and minimum.

Remember to analyze a function with absolute values it is best to write it as a piecewise defined function.

$$f(x) = \begin{cases} -3x - x^2 - 2 & -1 \leq x \leq 0 \\ 3x - x^2 - 2 & 0 \leq x \leq 2 \end{cases}$$

We must find the critical points of $f(x)$ in the interval $[-1, 2]$. $f'(x)$ does not exist at $x = 0$ since $|x|$ is not differentiable at $x = 0$.

Hence $x = 0$ is a critical point.

In the interval $[-1, 2]$, $f'(x) = 0$ if $-3 - 2x = 0$ for $-1 < x < 0$ or $3 - 2x = 0$ for $0 < x < 2$.

$-3 - 2x = 0$ if $x = -3/2$, however this is not a value of x with $-1 < x < 0$, so it is not a critical point in our interval.

$3 - 2x = 0$ if $x = 3/2$ and this is a value of x with $0 < x < 2$. Hence this is a critical point.

We check the function values at the critical points and at the end points of the interval:

$f(0) = -2$ MIN

$f(-1) = 0$

$f(3/2) = 1/4$ MAX

$f(2) = 0$

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11.(10 pts.) At noon ship A is 8 km west from ship B. Ship A is sailing south at 4 km/h and ship B is sailing north at 2km/h. How fast is the distance between the ships changing at 1p.m.?

Solution: Let $P(t)$ be the horizontal distance in km between the two ships. (Notice P is constant.) Let $Q(t)$ be the vertical distance in km between the two ships at a give time. Let $R(t)$ be the actual, diagonal, distance between the two ships at a given time. Measure time in hours beginning from noon. $R'(1)$ is the value we are asked to find. Notice that $P(t), Q(t)$, and $R(t)$ at a given time t are the lengths of the sides of a right triangle, so that

$$P(t)^2 + Q(t)^2 = R(t)^2.$$

$$R(t) = \sqrt{P(t)^2 + Q(t)^2}$$

We can plug in 8 km for $P(t)$.

$$R(t) = \sqrt{64 + Q(t)^2}$$

Differentiating with respect to t,

$$R'(t) = \frac{2QQ'(t)}{2\sqrt{64 + Q^2(t)}} = \frac{QQ'(t)}{\sqrt{64 + Q^2(t)}}.$$

We know $Q'(1) = 6$ km/h, since the ships are moving away from each other at a constant rate of 4 km/h + 2 km/h. $Q(1)$ is how far apart the ships are vertically at 1 p.m., which is 6 km. Therefore,

$$R'(1) = \frac{6 * 6}{\sqrt{64 + 6^2}} = \frac{36}{10} = 3.6\text{km/hr}.$$

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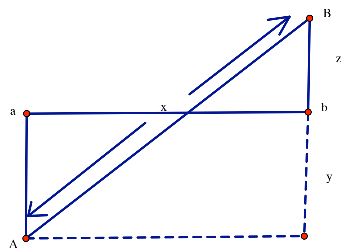
Second Solution



In the diagram above, a denotes the position of ship A at noon and A denotes the position of ship A at time t. Also b denotes the position of ship B at noon and B denotes the position of Ship B at time t.

Let x be the distance between the ships at time t, Let z be the distance travelled by ship B at time t and let y be the distance travelled by Ship A at time t.

We are given that $\frac{dz}{dt} = 2km/hr$ and $\frac{dy}{dt} = 4km/hr$ and the distance between a and b is 8km. . We wish to find $\frac{dx}{dt}$ when $z = 2$ and $y = 4$.

We have $x^2 = 8^2 + (y+z)^2$. Using implicit differentiation, we get $2x\frac{dx}{dt} = 2(y+z)(\frac{dy}{dt} + \frac{dz}{dt})$. This gives

$$\frac{dx}{dt} = \frac{2(y+z)(\frac{dy}{dt} + \frac{dz}{dt})}{2x} = \frac{2(y+z)(4+2)}{2x} = \frac{12(y+z)}{2x}.$$

When $z = 2$, $y = 4$, $x = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$. Therefore when $z = 2$, $y = 4$, $\frac{dx}{dt} = \frac{12(6)}{20} = 3.6$.

We conclude that at $1p.m.$ $\frac{dx}{dt} = 3.6km/hr$ or the distance between the ships is increasing at $3.6km/hr$.

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12.(10 pts.) Show that

$$2x - \sin(x) + x^3 + 2 = 0$$

has one and exactly one solution. Identify the theorem(s) you are using.

Solution: We will use the Intermediate Value Theorem to show there exists at least one solution and Rolle's Theorem to show there exists at most one solution. Let $f(x) = 2x - \sin(x) + x^3 + 2$.

Note that $f(0) = 2$ and $f(-\pi) = -\pi^3 - 2\pi + 3 < 0$. Since $f(-\pi) < 0 < f(0)$ there exists at least one solution in the interval $(-\pi, 0)$ by the Intermediate Value Theorem.

Next, $f'(x) = 2 - \cos(x) + 3x^2$, which is bounded below by $2 - 1 + 3(0^2) = 1$. So $f'(x) \geq 1$ for all x-values. If there were more than one solution then according to Rolle's Theorem there is some x such that $f'(x) = 0$, but that contradicts the fact that $f'(x) \geq 1$ everywhere. So there exists at most one solution.

We conclude that there exists exactly one solution to f.

Second Solution

We first use the Intermediate Value Theorem to show that $f(x) = 2x - \sin x + x^3 + 2 = 0$ has a solution. This function is continuous everywhere, so it is legal use the intermediate value theorem.

We have $f(0) = 2 > 0$ and $f(-2\pi) = 2(-2\pi) - (0) + (-2\pi)^3 + 2 = -4\pi - 8\pi^3 + 2 < 0$ since $\pi \approx 3.14$

By the Internediate Value Theorem, there is a number c with $-2\pi < c < 0$ for which $f(c) = 0$. Hence we know that there is at least one solution to the equation $f(x) = 0$.

Now let us assume that

There is another number $d \neq c$ for which $f(d) = 0$.

This assumption will lead us to a contradiction and we can then conclude that is is a false assumption:

If there is such a number d, then $f(d) = f(c) = 0$. We can apply Rolles theorem in the interval $[c, d]$ (or $[d, c]$ as appropriate), because $f(x) = 2x - \sin x + x^3 + 2$ is continuous in the interval $[c, d]$ and differentiable on (c, d). This tells us that there must be a value of x with $c < x < d$ with $f'(x) = 0$. However $f'(x) = 2 - cosx + 3x^2 \geq 1$ (since $\cos x < 1$ and $3x^2 \geq 0$. Therefore we have a contradiction and our assumption that there is a second solution must be false.

We conclude that there is only one solution to the equation $2x - \sin(x) + x^3 + 2 = 0$.

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