1. Solving the equation $3x - \cos x = 0$ using the Newton’s method with initial approximation $x_1 = 0$, what is $x_2$?

Solution: Let $f(x) = 3x - \cos x$. We have $f(0) = -1$, and since $f'(x) = 3 + \sin x$ we have $f'(0) = 3$. According to Newton’s Method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}.$$

2. A farmer has 4000 feet of fencing and wants to fence off a rectangular field that border a straight river. No fence is needed along the river. Find the dimension of this rectangle that will maximize the area.

Solution: Let $x$ be the length and $y$ be the width of the field. So we have

$$x + 2y = 4000 \iff y = \frac{1}{2}(4000 - x).$$

Consequently we have area given by

$$xy = x \cdot \frac{4000 - x}{2} = \frac{1}{2}(4000x - x^2).$$

Let’s denote the area function by $A(x)$. Since we need to maximize area we first differentiate $A(x)$ with respect to $x$:

$$A'(x) = \frac{1}{2}(4000 - 2x)$$

and then setting $A'(x) = 0$ gives $x = 2000$. Since $A(x)$ is a downward-opening parabola, we know that $x = 2000$ must be a maximum. The corresponding $y$-value is $\frac{1}{2}(4000 - 2000) = 1000$. Hence, the dimensions of the field are $2000 ft \times 1000 ft$.

3. Calculate the following indefinite integral

$$\int \frac{x^2 + 1}{\sqrt{x}} \, dx =$$

Solution:

$$\int \frac{x^2 + 1}{\sqrt{x}} \, dx = \int (x^{\frac{3}{2}} + x^{-\frac{1}{2}}) \, dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$$
4. Calculate the following definite integral
\[ \int_{0}^{3} |x - 1| \, dx = \]

**Solution:** The graph of \( y = |x-1| \) shows that the region between \( |x-1| \)
and the \( x \)-axis from 0 to 3 is composed of two triangles: let’s say \( \Delta_1 \)
with coordinates (0, 0), (0, 1), and (1, 0) and \( \Delta_2 \) with coordinates (1, 0),
(3, 0), and (3, 2). So using the formula \( \text{area}(\Delta) = \frac{1}{2}(\text{base})(\text{height}) \) we obtain
\[ \int_{0}^{3} |x - 1| \, dx = \text{area}(\Delta_1) + \text{area}(\Delta_2) \]
\[ = \frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) \]
\[ = \frac{1}{2} + 2 \]
\[ = \frac{5}{2}. \]

5. What is the indefinite integral
\[ \int x\sqrt{x - 1} \, dx = ? \]

**Solution:** Let \( u = x - 1; \) so \( du = dx \) and \( x = u + 1. \) Substituting
these into the integral we obtain
\[ \int (u + 1)\sqrt{u} \, du = \int u^{\frac{3}{2}} + u^{\frac{1}{2}} \, du \]
\[ = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C \]
\[ = \frac{2}{5}(x - 1)^{\frac{5}{2}} + \frac{2}{3}(x - 1)^{\frac{3}{2}} + C. \]

6. The equation of the slant asymptote of the curve \( y = \frac{3x^3 + 2x^2 + x + 3}{x^2 + 2x} \) is:

**Solution:** Carrying out long-division we obtain
\[ y = \frac{3x^3 + 2x^2 + x + 3}{x^2 + 2x} = 3x - 4 + \frac{9x + 3}{x^2 + 2x}. \]
Therefore, the slant asymptote is \( y = 3x - 4. \)
7. Evaluate the definite integral

\[ \int_0^2 \sqrt{4-x^2} \, dx. \]

**Solution:** Note that the given integral represents the area of the top-right quarter of the circle centered at the origin with radius 2. So we have

\[ \int_0^2 \sqrt{4-x^2} \, dx = \frac{1}{4} \pi (2)^2 = \pi. \]

8. Let \( g(x) = \int_0^{x^3} \sqrt{1 + \sin^2 t} \, dt. \) Find \( g'(x). \)

**Solution:** Let \( u = x^3. \) Then the Fundamental Theorem of Calculus gives

\[ g'(x) = \left( \frac{d}{du} \int_0^u \sqrt{1 + \sin^2 t} \, dt \right) \cdot \frac{du}{dx} = \sqrt{1 + \sin^2 (x^3)} \cdot 3x^2. \]

9. Calculate the area bounded by the curves \( y = x^2 + 2x + 3 \) and \( y = 2x + 4 \)

**Solution:** Let us first find the points of intersection of the two functions:

\[ x^2 + 2x + 3 = 2x + 4 \iff x^2 - 1 = 0 \iff x = \pm 1. \]

Note that the line lies above the parabola on the interval \((-1, 1)\) (check by plugging in \( x = 0 \), for example), so the area of the bounded region is given by the following definite integral:

\[ \int_{-1}^{1} [(2x + 4) - (x^2 + 2x + 3)] \, dx = \int_{-1}^{1} (-x^2 + 1) \, dx = \left[ -\frac{x^3}{3} + x \right]_{-1}^{1} \]
\[ = \left( \frac{(-1)^3}{3} + 1 \right) - \left( \frac{1^3}{3} - 1 \right) = \left( \frac{2}{3} \right) - \left( -\frac{2}{3} \right) = \frac{4}{3}. \]
10. (a) Evaluate the definite integral \( \int_0^2 x^2 \, dx \) by the limit definition.

Hint: \( 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)

(b) Verify your result using the fundamental theorem of calculus.

Solution: (a) [Aside: Note that we can use either right endpoints or left endpoints here (since \( n \to \infty \) in the limit definition), the only difference being that in the former case we would be summing from \( i = 1 \) to \( n \) and in the latter case we would sum from \( i = 0 \) to \( n - 1 \). Given that \( n \) is simpler to work with in the hint (rather than \( n - 1 \)), we will use right endpoints.]

We need to work with the limit definition of the definite integral:

\[
\int_0^2 x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,
\]

where \( f(x) = x^2 \), \( \Delta x = \frac{2-0}{n} = \frac{2}{n} \) and \( x_i = 0 + i \Delta x = \frac{2i}{n} \). Let us first obtain a simpler version of the summation:

\[
\sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} \left( \frac{2i}{n} \right)^2 \left( \frac{2}{n} \right) = \sum_{i=1}^{n} \frac{8}{n^3} \cdot i^2 = \frac{8}{n^3} \sum_{i=1}^{n} i^2 \]

\[
= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \]

\[
= \frac{4}{3} \left( \frac{(n^2 + n)(2n + 1)}{n^3} \right) \]

\[
= \frac{4}{3} \left( \frac{2n^3 + 3n^2 + n}{n^3} \right)
\]

We now evaluate the limit of the sum as \( n \) approaches infinity:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} \left( \frac{4}{3} \cdot \frac{2n^3 + 3n^2 + n}{n^3} \right) \]

\[
= \frac{4}{3} \cdot \lim_{n \to \infty} \left( \frac{2n^3 + 3n^2 + n}{n^3} \right) \]

\[
= \frac{4}{3} \cdot 2 = \frac{8}{3}
\]

(b) \( \int_0^2 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3} \)
11. Find the point on the line $3x + y = 9$ that is closest to the point $(1, -2)$.

**Solution:** Let $(x, y)$ denote a point on the line $y = -3x + 9$; so $(x, y) = (x, -3x + 9)$. The distance between points $(x, -3x + 9)$ and $(1, -2)$ is given by

$$d(x) = \sqrt{(x - 1)^2 + ((-3x + 9) - (-2))^2} = \sqrt{10x^2 - 68x + 122}.$$  

To minimize distance, we differentiate $d(x)$ with respect to $x$:

$$d'(x) = \frac{1}{2}(10x^2 - 68x + 122)^{-\frac{1}{2}}(20x - 68) = \frac{10x - 34}{\sqrt{10x^2 - 68x + 122}}.$$  

Setting $d'(x) = 0$ gives $x = 3.4$. An easy way to verify that $x$ is indeed a minimum is to observe that $d(x)$ is the square root of an upward-opening parabola. The corresponding $y$-value is given by $y = -3(3.4) + 9 = -10.2 + 9 = -1.2$.

12. If 1200 cm$^2$ of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

**Solution:** Let $x$ denote the length of a side of the square base and $y$ denote the height of the box. Then the surface area of the open box is given by

$$x^2 + 4xy = 1200 \iff y = \frac{1200 - x^2}{4x}.$$  

The volume of the box is given by

$$x^2y = x^2 \cdot \frac{1200 - x^2}{4x} = 300x - \frac{1}{4}x^3.$$  

Let us denote the volume function by $V(x)$. Since we need to maximize the volume over the interval $(0, \infty)$, we differentiate $V(x)$ with respect to $x$:

$$V'(x) = 300 - \frac{3}{4}x^2.$$  

Setting $V'(x) = 0$ we get $x = 20$. To check that $x = 20$ gives us a maximum, let us take another derivative: $V''(x) = -\frac{3}{2}x$; since $V''(20) = -30 < 0$ we have confirmed (by the Second Derivative Test) that $x = 20$ is local maximum, but since this is the only critical point in $(0, \infty)$, it is also a global/absolute maximum. Consequently, the largest possible volume of the box is

$$V(20) = 400 \cdot \frac{1200 - 400}{80} \text{cm}^3 = 4000 \text{cm}^3.$$