MATH 10550, PRACTICE EXAM 3 SOLUTIONS

1. *Solving the equation* $3x - \cos x = 0$ *using the Newton's method with initial approximation* $x_1 = 0$, *what is* x_2?

Solution: Let $f(x) = 3x - \cos x$. We have $f(0) = -1$, and since $f'(x) = 3 + \sin x$ we have $f'(0) = 3$. According to Newton's Method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}.$$

2. *A farmer has 4000 feet of fencing and wants to fence off a rectangular field that border a straight river. No fence is needed along the river. Find the dimension of this rectangle that will maximize the area.*

Solution: Let x be the length and y be the width of the field. So we have

$$x + 2y = 4000 \Longleftrightarrow y = \frac{1}{2}(4000 - x).$$

Consequently we have area given by

$$xy = x \cdot \frac{4000 - x}{2} = \frac{1}{2}(4000x - x^2).$$

Let's denote the area function by $A(x)$. Since we need to maximize area we first differentiate $A(x)$ with respect to x:

$$A'(x) = \frac{1}{2}(4000 - 2x)$$

and then setting $A'(x) = 0$ gives $x = 2000$. Since $A(x)$ is a downward-opening parabola, we know that $x = 2000$ must be a maximum. The corresponding y-value is $\frac{1}{2}(4000 - 2000) = 1000$. Hence, the dimensions of the field are $2000ft \times 1000ft$.

3. *Calculate the following indefinite integral*

$$\int \frac{x^2 + 1}{\sqrt{x}}\, dx =$$

Solution:

$$\int \frac{x^2 + 1}{\sqrt{x}}\, dx = \int (x^{\frac{3}{2}} + x^{-\frac{1}{2}})\, dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$$

4. *Calculate the following definite integral*

$$\int_0^3 |x-1|\,dx =$$

Solution: The graph of $y = |x-1|$ shows that the region between $|x-1|$ and the x-axis from 0 to 3 is composed of two triangles: let's say Δ_1 with coordinates $(0,0)$, $(0,1)$, and $(1,0)$ and Δ_2 with coordinates $(1,0)$, $(3,0)$, and $(3,2)$. So using the formula $area(\Delta) = \frac{1}{2}(base)(height)$ we obtain

$$\begin{aligned}\int_0^3 |x-1|\,dx &= area(\Delta_1) + area(\Delta_2)\\ &= \frac{1}{2}(1)(1) + \frac{1}{2}(2)(2)\\ &= \frac{1}{2} + 2\\ &= \frac{5}{2}.\end{aligned}$$

5. *What is the indefinite integral*

$$\int x\sqrt{x-1}\,dx =?$$

Solution: Let $u = x-1$; so $du = dx$ and $x = u+1$. Substituting these into the integral we obtain

$$\begin{aligned}\int (u+1)\sqrt{u}\,du &= \int u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du\\ &= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C.\end{aligned}$$

6. *The equation of the slant asymptote of the curve* $y = \frac{3x^3+2x^2+x+3}{x^2+2x}$ *is:*
Solution: Carrying out long-division we obtain

$$y = \frac{3x^3+2x^2+x+3}{x^2+2x} = 3x - 4 + \frac{9x+3}{x^2+2x}.$$

Therefore, the slant asymptote is $y = 3x - 4$.

7. *Evaluate the definite integral*

$$\int_0^2 \sqrt{4-x^2}\,dx.$$

Solution: Note that the given integral represents the area of the top-right quarter of the circle centered at the origin with radius 2. So we have

$$\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi(2)^2 = \pi.$$

8. *Let* $g(x) = \int_0^{x^3} \sqrt{1+\sin^2 t}\,dt$. *Find* $g'(x)$.

Solution: Let $u = x^3$. Then the Fundamental Theorem of Calculus gives

$$g'(x) = \left(\frac{d}{du}\int_0^u \sqrt{1+\sin^2 t}\,dt\right)\cdot\frac{du}{dx} = \sqrt{1+\sin^2(x^3)}\cdot\ 3x^2.$$

9. *Calculate the area bounded by the curves* $y = x^2+2x+3$ *and* $y = 2x+4$

Solution: Let us first find the points of intersection of the two functions:

$$x^2+2x+3 = 2x+4 \iff x^2-1=0 \iff x=\pm 1.$$

Note that the line lies above the parabola on the interval $(-1,1)$ (check by plugging in $x=0$, for example), so the area of the bounded region is given by the following definite integral:

$$\int_{-1}^1 [(2x+4)-(x^2+2x+3)]\,dx = \int_{-1}^1 (-x^2+1)\,dx = \left[-\frac{x^3}{3}+x\right]_{-1}^1$$
$$= \left(\frac{(-1)^3}{3}+1\right)-\left(\frac{1^3}{3}-1\right) = \left(\frac{2}{3}\right)-\left(-\frac{2}{3}\right) = \frac{4}{3}$$

10. *(a) Evaluate the definite integral* $\displaystyle\int_0^2 x^2\,dx$ *by the limit definition.*

Hint: $1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(b) Verify your result using the fundamental theorem of calculus.

Solution: (a) [Aside: Note that we can use either right endpoints or left endpoints here (since $n \to \infty$ in the limit definition), the only difference being that in the former case we would be summing from $i = 1$ to n and in the latter case we would sum from $i = 0$ to $n - 1$. Given that n is simpler to work with in the hint (rather than $n - 1$), we will use right endpoints.]

We need to work with the limit definition of the definite integral:

$$\int_0^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $f(x) = x^2$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ and $x_i = 0 + \Delta x \cdot i = \frac{2i}{n}$. Let us first obtain a simpler version of the summation:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{8}{n^3} \cdot i^2 = \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{4}{3}\left(\frac{(n^2+n)(2n+1)}{n^3}\right) \\ &= \frac{4}{3}\left(\frac{2n^3+3n^2+n}{n^3}\right)\end{aligned}$$

We now evaluate the limit of the sum as n approaches infinity:

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \left(\frac{4}{3} \cdot \frac{2n^3+3n^2+n}{n^3}\right) \\ &= \frac{4}{3} \cdot \left(\lim_{n\to\infty} \frac{2n^3+3n^2+n}{n^3}\right) \\ &= \frac{4}{3} \cdot 2 = \frac{8}{3}\end{aligned}$$

(b) $\displaystyle\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$

11. *Find the point on the line $3x + y = 9$ that is closest to the point $(1, -2)$.*

Solution: Let (x, y) denote a point on the line $y = -3x + 9$; so $(x, y) = (x, -3x + 9)$. The distance between points $(x, -3x + 9)$ and $(1, -2)$ is given by

$$d(x) = \sqrt{(x-1)^2 + ((-3x+9) - (-2))^2} = \sqrt{10x^2 - 68x + 122}.$$

To minimize distance, we differentiate $d(x)$ with respect to x:

$$d'(x) = \frac{1}{2}(10x^2 - 68x + 122)^{-\frac{1}{2}}(20x - 68) = \frac{10x - 34}{\sqrt{10x^2 - 68x + 122}}.$$

Setting $d'(x) = 0$ gives $x = 3.4$. An easy way to verify that x is indeed a minimum is to observe that $d(x)$ is the square root of an upward-opening parabola. The corresponding y-value is given by $y = -3(3.4) + 9 = -10.2 + 9 = -1.2$.

12. *If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.*

Solution: Let x denote the length of a side of the square base and y denote the height of the box. Then the surface area of the open box is given by

$$x^2 + 4xy = 1200 \iff y = \frac{1200 - x^2}{4x}.$$

The volume of the box is given by

$$x^2 y = x^2 \cdot \frac{1200 - x^2}{4x} = 300x - \frac{1}{4}x^3.$$

Let us denote the volume function by $V(x)$. Since we need to maximize the volume over the interval $(0, \infty)$, we differentiate $V(x)$ with respect to x:

$$V'(x) = 300 - \frac{3}{4}x^2.$$

Setting $V'(x) = 0$ we get $x = 20$. To check that $x = 20$ gives us a maximum, let us take another derivative: $V''(x) = -\frac{3}{2}x$; since $V''(20) = -30 < 0$ we have confirmed (by the Second Derivative Test) that $x = 20$ is local maximum, but since this is the only critical point in $(0, \infty)$, it is also a global/absolute maximum. Consequently, the largest possible volume of the box is

$$V(20) = 400 \cdot \frac{1200 - 400}{80} cm^3 = 4000 cm^3.$$