Name: ______________________________

Instructor: __________________________

Math 10550, Exam I
September 25, 3024

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-15.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(e)
8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)
11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	______________
6.	______________
13.	______________
14.	______________
15.	______________
Total	______________

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Multiple Choice

1.(7 pts.) Compute
$$\lim_{x\to 2} \frac{\sqrt{x^2+21}-5}{x-2}.$$

Solution:
$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2+21}-5}{x-2} &= \lim_{x\to 2} \frac{\sqrt{x^2+21}-5}{x-2}\frac{\sqrt{x^2+21}+5}{\sqrt{x^2+21}+5} \\
&= \lim_{x\to 2} \frac{x^2-4}{(x-2)(\sqrt{x^2+21}+5)} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+21}+5)} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+21}+5} \\
&= \frac{4}{\sqrt{25}+5} \\
&= \frac{4}{10} \\
&= \frac{2}{5}.
\end{aligned}$$

(a) $\frac{2}{5}$ (b) 0 (c) $\frac{1}{10}$ (d) $\frac{1}{120}$ (e) 4

2.(7 pts.) Compute
$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(5x)}.$$

Solution: If we try to just "plug in" then we get
$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(5x)} = \frac{0}{0},$$
which is hogwash.
$$\lim_{x\to 0} \frac{\sin(2x)}{\sin(5x)} = \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{5x}{\sin(5x)} \cdot \frac{2x}{5x} = \frac{2}{5}$$
since $\lim_{y\to 0} \frac{\sin y}{y} = 1$.

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(a) $\frac{5}{2}$

(b) 0

(c) $\frac{2}{5}$

(d) 1

(e) Does not exist.

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3.(7 pts.) Compute $\lim_{x \to -1^-} \dfrac{x^2+x}{x^2+2x+1}$

(a) $-\infty$

(b) -1

(c) 0

(d) Does not exist and is not ∞ or $-\infty$.

(e) $+\infty$

Solution: First, note that $\lim_{x \to -1^-} \dfrac{x^2+x}{x^2+2x+1} = \lim_{x \to -1^-} \dfrac{x(x+1)}{(x+1)(x+1)} = \lim_{x \to -1^-} \dfrac{x}{x+1}$. Now if you plug in $x=-1$ you get $\dfrac{-1}{0}$. Now, since we are looking at the limit from the left, we must argue if the limit goes to $+\infty$ or $-\infty$. Indeed the limit of the numerator is the constant -1. Additionally, the when approaching -1 from the left, the denominator takes on negative values closer and closer to 0. So $\lim_{x \to -1^-} \dfrac{x^2+x}{x^2+2x+1} = +\infty$.

4.(7 pts.) For what values of c is the function f given by

$$f(x) = \begin{cases} x^2 + c^2x - 3 & x < 2 \\ \\ cx + 5 & x \geq 2 \end{cases}$$

continuous at $x = 2$?

(a) $c = 1$ only

(b) $c = 2$ and $c = -1$

(c) $c = 0$ only

(d) No value of c makes f continuous at $x = 2$

(e) $c = 2$ only

Solution: In order for $f(x)$ to be continuous at $x = 2$ we need

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2).$$

We have $\lim_{x \to 2^-} f(x) = 2^2 + 2c^2 - 3$ and $f(2) = 2c + 5$, and so $2^2 + 2c^2 - 3 = 2c + 5$. Therefore $c = 2, -1$.

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5.(7 pts.) Let $f(x) = \sqrt{2x^2+1}$. Which of the following limits equals $f'(2)$?

(a) $\lim\limits_{h\to 2} \dfrac{\sqrt{2(x+h)^2+1}-3}{h}$

(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{2x^2+1}-3}{x}$

(c) $\lim\limits_{x\to 2} \dfrac{\sqrt{2x^2+1}-3}{x-2}$

(d) $\lim\limits_{h\to 2} \dfrac{\sqrt{2(x+h)^2+1}-\sqrt{2x^2+1}}{h}$

(e) $\lim\limits_{h\to 0} \dfrac{\sqrt{2(x+h)^2+1}-\sqrt{2x^2+1}}{h}$

Solution: Note that for any a we have $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$. Now, $f(2) = \sqrt{2(2^2)+1} = \sqrt{9} = 3$. So $f'(2) = \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 2} \dfrac{\sqrt{2x^2+1}-3}{x-2}$.

6.(7 pts.) Assume that $f(x)$ is a continuous function which takes the following values:

x	-1	0	1	2
$f(x)$	-10	10	-1	3

Which of the following conclusions can we make by using the Intermediate Value Theorem:

Solution: We see that $f(-1) < 0 < f(0)$ and hence there is $x_1 \in (-1,0)$ such that $f(x_1) = 0$ by the Intermediate Value Theorem (IVT) since f is continuous. Further $f(0) > 0 > f(1)$ and hence there is $x_2 \in (0,1)$ such that $f(x_2) = 0$ by IVT. Lastly $f(1) < 0 < f(2)$ and hence there is $x_3 \in (1,2)$ such that $f(x_3) = 0$ by IVT. We can conclude that there are at least 3 three solutions to $f(x) = 0$ (note that there could be more).

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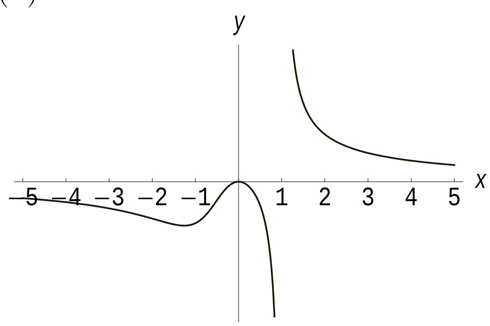
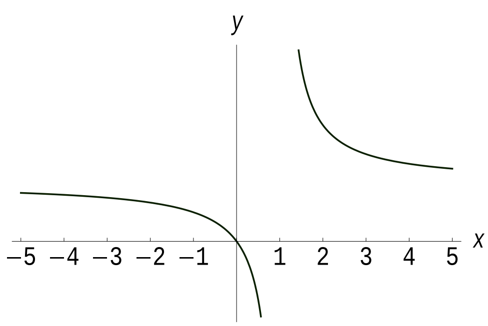
7.(7 pts.) The graph of $f(x)$ is shown below:

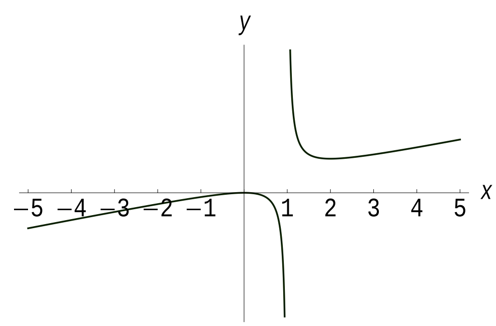


Which of the following is the graph of $f'(x)$?

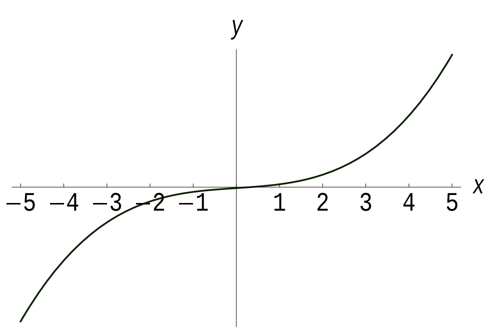
(a)



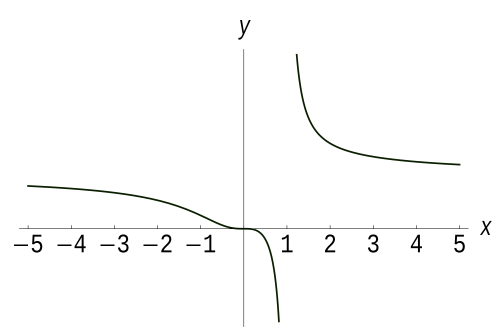
(b)



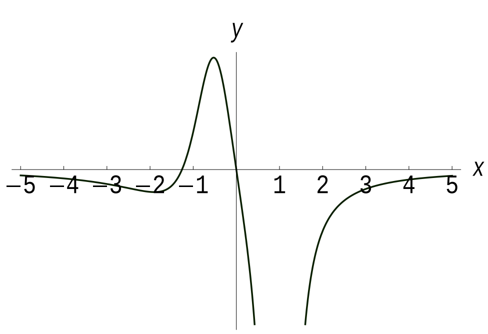
(c)



(d)



(e)



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Solution: We observe that $f(x)$ has two horizontal tangent lines. One has the $x-$coordinate located somewhere in the interval [-2,-1] and the other one at $x = 0$. Hence $f'(x)$ has two zeroes at these two locations. Only graph (e) has this property.

8.(7 pts.) Find $f'(x)$, if

$$f(x) = 2x^2 \sin(\sqrt{x}) + \frac{1}{\sqrt{x}}.$$

(a) $4x \cos(\sqrt{x}) - \dfrac{1}{2\sqrt{x^3}}$

(b) $\sqrt{x^3} \cos(\sqrt{x}) + 4x \sin(\sqrt{x}) - \dfrac{1}{2\sqrt{x^3}}$

(c) $-\sqrt{x^3} \cos(\sqrt{x}) + \sin(\sqrt{x}) + \dfrac{1}{2\sqrt{x^3}}$

(d) $2x^2 \cos(\sqrt{x}) + 4x \sin(\sqrt{x}) - \dfrac{1}{2\sqrt{x^3}}$

(e) $-\sqrt{x^3} \cos(\sqrt{x}) + 4x \sin(\sqrt{x}) - \dfrac{1}{2\sqrt{x^3}}$

Solution: Recall that the derivative of a sum is equal to the sum of the derivatives. The derivative of the first summand $2x^2 \sin(\sqrt{x})$ requires the product rule and the chain rule. Indeed, the derivative is $2x^2 \frac{d}{dx}(\sin(\sqrt{x}) + \frac{d}{dx}(2x^2)\sin(\sqrt{x}) = 2x^2 \cos(\sqrt{x})(1/2)x^{-1/2} + 4x \sin(\sqrt{x}) = \sqrt{x^3}\cos(\sqrt{x}) + 4x\sin(\sqrt{x})$.

The derivative of the second summand $\frac{1}{\sqrt{x}}$ simply requires the power rule and is equal to $-\frac{1}{2\sqrt{x^3}}$.

Thus the derivative of $f(x) = \sqrt{x^3}\cos(\sqrt{x}) + 4x\sin(\sqrt{x}) - \dfrac{1}{2\sqrt{x^3}}$

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9.(7 pts.) Find the derivative of $f(x) = \tan(\sin(x^2))$.

(a) $\cot(\sin(x^2))\cos(x^2))$

(b) $-2x\sec^2(\sin(x^2))\cos(x^2))$

(c) $2x\sec^2(\sin(x^2))\cos(x^2))$

(d) $2x\sec^2(\sin(x^2))\sin(x^2)$

(e) $2x\cot(\sin(x^2))\cos(x^2))$

Solution. Let $f_1(x) = \tan x$, $f_2(x) = \sin x$, $f_3(x) = x^2$. We have $f_1'(x) = \sec^2(x)$, $f_2'(x) = \cos x$, $f_3'(x) = 2x$. Note $f(x) = f_1(f_2(f_3(x)))$. By chain rule, $f'(x) = f_1'(f_2(f_3(x)))f_2'(f_3(x))f_3'(x) = 2x\sec^2(\sin x^2)\cos(x^2)$.

10.(7 pts.) If $f(x) = x\sin x + \cos x$, find $f''(x)$.

(a) $f''(x) = x\cos x + \sin x$

(b) $f''(x) = -x\sin x - \cos x$

(c) $f''(x) = 3\cos x - x\sin x$

(d) $f''(x) = -x\sin x + \cos x$

(e) $f''(x) = -\sin x - \cos x$

Solution:
First note that $f'(x) = x\cos x + \sin x - \sin x = x\cos x$. Then $f''(x) = x(-\sin x) + \cos x$.

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11.(7 pts.) Let $h(x) = f \circ g(x) - \dfrac{f(x)}{g(x)}$. If $f(3) = 0$, $g(3) = 1$, $f'(3) = 3$, $g'(3) = 4$, $f'(1) = 7$, and $g'(2) = 5$, then find $h'(3)$.

(a) 30 (b) 20 (c) 25 (d) 10 (e) 0

Solution: First, note that for any x, we have $h'(x) = f'(g(x))g'(x) - \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.
Therefore $h'(3) = f'(g(3))g'(3) - \dfrac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} = f'(1)g'(3) - \dfrac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} =$
$7 * 4 - \dfrac{1 * 3 - 0 * 4}{1^2} = 28 - 3 = 25$

12.(7 pts.) If $f(x) = x^3 - 3x^2 - 9x + 7$, find the x-coordinates of all points on the curve with horizontal tangent line.

(a) $x = 0$ and $x = 1$

(b) $x = 3$ and $x = -1$

(c) $x = 4$ and $x = -2$

(d) $x = -3$ and $x = 1$

(e) No points on the curve have horizontal tangent line.

Solution: We solve $f'(x) = 3x^2 - 6x - 9 = 0$. So $x = 3, -1$.

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Partial Credit

You must show your work on the partial credit problems to receive credit!

13.(10 pts.) Find the derivative of

$$f(x) = \sqrt{x+1}$$

using the limit definition of the derivative.

Please include all of the details in your calculation.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} * \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}
\end{aligned}$$

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14.(10 pts.) Let $y = x^2 + x$.

(a) Find the equation of the tangent line through the point $(-1, 0)$.

Solution: $y'(-1) = (x^2+x)'|_{x=-1} = 2x+1|_{x=-1} = -1$. Hence the slope of the tangent line of $y = x^2 + x$ is -1 at $(-1, 0)$. So the equation is $y = -(x + 1) = -x - 1$.

(b) Find all points on the curve whose tangent line goes through the point $(2, 5)$.

Solution: Suppose such a point has coordinate $(a, a^2 + a)$. Then the slope of the tangent line at that point is $y'(a) = 2a + 1$. On the other hand this slope is given by $\frac{a^2 + a - 5}{a - 2}$. Therefore $2a + 1 = \frac{a^2 + a - 5}{a - 2}$. So $a = 1$ or 3. Such points can be $(1, 2)$ or $(3, 12)$.

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15.(10 pts.) Show that there is at least one solution of the equation

$$x^2 = 2 + \sin(\pi x).$$

Justify your answer, identify the theorem you use and explain why the theorem applies.

Solution:

First, note that finding a solution to $x^2 = 2 + \sin(\pi x)$ is equivalent to finding the zeros to the function $f(x) = 2 + \sin(\pi x) - x^2$. Indeed, $f(x)$ is continuous since 2 ,$\sin(\pi x)$ and $-x^2$ are all continuous and the sum of continuous functions is again continuous.

Further note that $f(0) = 2 + \sin(0) - 0^2 = 2$ and $f(2) = 2 + \sin(2\pi) - 4 = -2$.

The intermediate value theorem states that for any continuous function on an interval $[a, b]$ and a number N between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$ there is a number $c \in (a, b)$ such that $f(c) = N$. We can apply the IVT to our case and conclude that since $f(2) = -2 < 0 < 2 = f(0)$ there is some $c \in (0, 2)$ such that $f(c) = 0$.

Indeed for that c, we will have $c^2 = 2 + \sin(c\pi)$.

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Rough Work

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3.	(a)	(b)	(c)	(d)	(•)
4.	(a)	(•)	(c)	(d)	(e)
5.	(a)	(b)	(•)	(d)	(e)
7.	(a)	(b)	(c)	(d)	(•)
8.	(a)	(•)	(c)	(d)	(e)
9.	(a)	(b)	(•)	(d)	(e)
10.	(a)	(b)	(c)	(•)	(e)
11.	(a)	(b)	(•)	(d)	(e)
12.	(a)	(•)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	________
6.	________
13.	________
14.	________
15.	________
Total	________