10550 PRACTICE FINAL EXAM SOLUTIONS

1. First notice that
$$\frac{x^2-4}{x^2-5x+6} = \frac{(x-2)(x+2)}{(x-2)(x-3)}.$$
This function is undefined at $x = 2$. Since, in the limit as $x \to 2^-$, we only care about what happens near $x = 2$ (an for x less than 2), we can cancel
$$\lim_{x\to 2^-} \frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2^-} \frac{x+2}{x-3} = \frac{4}{-1} = -4.$$

2. As $x \to 0^+$, the numerator of our expression approaches -9, while the denominator approaches 0. This implies that this limit cannot be finite. The only remaining question is whether the answer is $\pm\infty$. When x is near 0, our denominator is near -9, i.e. it is negative. When x is near zero and greater than zero, $\sin x$ is near zero and positive. Hence for x near zero and positive, the expression is negative, and so the answer is $-\infty$.

3. We multiply the expression the limit by its conjugate to see that
$$\begin{aligned}\left(\sqrt{x^2-x}-\sqrt{x^2+5x}\right)\left(\frac{\sqrt{x^2-x}+\sqrt{x^2+5x}}{\sqrt{x^2-x}+\sqrt{x^2+5x}}\right) &= \frac{(x^2-x)-(x^2+5x)}{\sqrt{x^2-x}+\sqrt{x^2+5x}} \\ &= \frac{-6x}{|x|\left(\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{5}{x}}\right)}.\end{aligned}$$
It follows that
$$\lim_{x\to\infty}\left(\sqrt{x^2-x}-\sqrt{x^2+5x}\right) = \lim_{x\to\infty} \frac{-6x}{|x|\left(\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{5}{x}}\right)} = \lim_{x\to\infty} \frac{-6}{\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{5}{x}}} = -3.$$

4. The function f as defined is differentiable on $(-\infty, 0)$ and $(0, \infty)$ for any value of a. We need to choose a such that f is differentiable at $x = 0$. We require that both the right and left-hand limits in the definition of derivative agree at $x = 0$. We compute
$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(h^2+1)-1}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0.$$
While,
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(ah+1)-1}{h} = \lim_{h\to 0^-} \frac{ah}{h} = a.$$
It follows that we require $a = 0$ if we want $f'(0)$ to exist.

5. We first note that
$$\frac{\tan 2x}{\sin 3x} = \left(\frac{1}{\cos 2x}\right)\left(\frac{\sin 2x}{\sin 3x}\right).$$
Since $\lim_{x\to 0} \cos 2x = 1$, it follows that
$$\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} = \left(\lim_{h\to 0} \frac{1}{\cos 2x}\right)\left(\lim_{h\to 0} \frac{\sin 2x}{\sin 3x}\right) = \lim_{h\to 0} \frac{\sin 2x}{\sin 3x}$$
To evaluate this last limit we note that
$$\frac{\sin 2x}{\sin 3x} = \left(\frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}}\right)\left(\frac{2x}{3x}\right)$$

now using the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$ (an more generally that $\lim_{x\to 0}\frac{\sin kx}{kx}=1$ for any k), we get that

$$\lim_{x\to 0}\frac{\tan 2x}{\sin 3x}=\left(\lim_{x\to 0}\frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}}\right)\left(\lim_{x\to 0}\frac{2x}{3x}\right)=\frac{2}{3}.$$

6. We first notice that

$$\sqrt{4x^2+x+1}=\sqrt{(4x^2)(1+\frac{1}{4x}+\frac{1}{4x^2})}=2|x|\sqrt{1+\frac{1}{4x}+\frac{1}{4x^2}}.$$

So it follows that

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{4x^2+x+1}}{3x-1}\ \lim_{x\to-\infty}\frac{2|x|\sqrt{1+\frac{1}{4x}+\frac{1}{4x^2}}}{3x(1-\frac{1}{3x})}\\ &=\left(\lim_{x\to-\infty}\frac{2|x|}{3x}\right)\left(\lim_{x\to-\infty}\frac{\sqrt{1+\frac{1}{4x}+\frac{1}{4x^2}}}{1-\frac{1}{3x}}\right)\\ &=\left(-\frac{2}{3}\right)(1)\\ &=-\frac{2}{3}.\end{aligned}$$

Here we've used the fact that when x is negative, $\frac{|x|}{x}=-1$.

7. We perform implicit differentiation on the equation

$$x^2+4y^2=5$$

to yield the equation

$$2x+8y\frac{dy}{dx}=0.$$

Now setting $x=1$ and $y=-1$ yields the equation

$$2-8\frac{dy}{dx}=0,$$

or $\frac{dy}{dx}=\frac{1}{4}$. This is the slope of the tangent line to the ellipse at $(1,-1)$. It follows that the tangent line has point-slope form

$$y+1=\frac{1}{4}(x-1)\Rightarrow y=\frac{1}{4}x-\frac{5}{4}.$$

8. Since $F(x)=f(g(x))$, the chain rule gives that

$$F'(x)=f'(g(x))g'(x).$$

So

$$F'(2)=f'(g(2))g'(2)=f'(-1)\times 5=2\times 5=10.$$

9. The chain rule gives

$$y'=8\left(\sin 4x\right)^7\frac{d}{dx}\left(\sin 4x\right)=32\left(\sin 4x\right)^7\cos 4x.$$

10. We compute the first two derivatives

$$y'=x^4+x^3$$

and

$$y''=4x^3+3x^2=x^2(4x+3).$$

Our candidates for inflection points are points where the second derivative is zero, which is at $x=0$ and $x=-\frac{3}{4}$. The quantity x^2 is always positive, while $4x+3$ is negative for $x<-\frac{3}{4}$ and positive for $x>-\frac{3}{4}$. It follows that our original function is concave down for $x<-\frac{3}{4}$ and concave up for $x>-\frac{3}{4}$. Hence there is only one inflection point.

11. We implicitly differentiate the expression

$$\sqrt{x^2+y^2} = 2+y$$

and get

$$\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\frac{dy}{dx}) = \frac{dy}{dx}.$$

Plugging in $x = 4$ and $y = 3$ gives

$$\frac{1}{2}(25)^{-1/2}(8+6\frac{dy}{dx}) = \frac{dy}{dx},$$

or

$$\frac{1}{10}(8+6\frac{dy}{dx}) = \frac{dy}{dx}$$

This equation has solution $\frac{dy}{dx} = 2$.

12. We consider a right triangle with base x and height 100. Let θ be the angle such that $\tan\theta = \frac{100}{x}$, or $x\tan\theta = 100$. Here x and θ are functions of time. When the hypotenuse of our triangle is 200, the Pythagorean Theorem implies that $x = 100\sqrt{3}$, and so $\theta = \frac{\pi}{6}$. To find the expression that relates our rates, we differentiate (with respect to t) the equation

$$x\tan\theta = 100$$

to get

$$\frac{dx}{dt}\tan\theta + x\sec^2\theta\frac{d\theta}{dt} = 0.$$

Plugging in $x = 100\sqrt{3}$, $\theta = \frac{\pi}{6}$, and $\frac{dx}{dt} = 16$, we get that

$$16\left(\frac{\sqrt{3}}{3}\right) + 100\sqrt{3}\left(\frac{4}{3}\right)\frac{d\theta}{dt} = 0,$$

or $\frac{d\theta}{dt} = -\frac{1}{25}$.

13. We note that

$$f'(x) = \frac{1}{2}\left(10-x^2\right)^{-1/2}(-2x),$$

and so $f'(-1) = \frac{1}{3}$. Also note that $f(-1) = 3$. It follows that the linearization of f at $a = -1$ is

$$L(x) = 3 + \frac{1}{3}(x+1).$$

14. We could rewrite f as

$$f(x) = \begin{cases} 2x - x^2 - 1 & x \geq 0 \\ -2x - x^2 - 1 & x < 0 \end{cases}$$

Note that f is not differentiable at $x = 0$ (since $|x|$ is not) and so f has a critical point at $x = 0$. For $x < 0$, we have

$$f'(x) = -2 - 2x,$$

which is zero at $x = -1$. For $x > 0$, we have

$$f'(x) = 2 - 2x,$$

which is zero at $x = 1$. So our critical points are $x = 0, -1, 1$. It follows from looking at the sign of our expression for f' above that f is increasing on $(-\infty, -1) \cup (0, 1)$, while f is decreasing on $(-1, 0) \cup (1, \infty)$. By the first derivative test, this implies that we have local maxes at $x = \pm 1$, and a local minimum at $x = 0$.

15. Note that $x = 1$ is not a root of the numerator and so we cannot factor out an $(x - 1)$ from the numerator to cancel that in the denominator. Hence our function has a vertical asymptote at $x = 1$. By polynomial long division we have

$$\frac{2x^2 + x + 1}{x - 1} = 2x + 3 + \frac{4}{x - 1},$$

and so our function has a slant asymptote of $y = 2x + 3$. Finally, since

$$\lim_{x \to \infty} \frac{2x^2 + x + 1}{x - 1} = \infty \qquad \lim_{x \to -\infty} \frac{2x^2 + x + 1}{x - 1} = -\infty$$

we have no horizontal asymptotes.

16. The distance of an arbitrary point (x, y) from the point $(2, 0)$ is given by

$$d = \sqrt{(x - 2)^2 + y^2}.$$

When our points are constrained to be on the hyperbola $y^2 - x^2 = 4$, we can make the substitution $y^2 = x^2 + 4$ to get

$$d(x) = \sqrt{2x^2 - 4x + 8}.$$

We differentiate the above to find that

$$d'(x) = \frac{2x - 2}{\sqrt{2x^2 - 4x + 8}},$$

and so d has a critical point when $x = 1$. This is a minimum, since it is clearly not a maximum (points can be arbitrarily far away from $(2, 0)$ on the hyperbola). So when $x = 1$, we have $y^2 = 5$, and so our two point where the distance is minimized are $(1, \pm\sqrt{5})$.

17. If we denote the dimensions of the page as h and w (for height and width), then we have that the printed area will have the formula

$$P = (w - 2)(h - 3).$$

We have the constraint

$$150 = hw$$

which allows us to make the substitution $w = \frac{150}{h}$, yielding

$$P(h) = (\frac{150}{h} - 2)(h - 3) = 156 - 2h - \frac{450}{h}.$$

Differentiating gives

$$P'(h) = -2 + \frac{450}{h^2}.$$

We note that a critical point occurs at $h = 15$. When $h = 15$, then $w = 10$. This is a maximum because the extremes ($h = 3$ and $w = 50$ or $h = 75$ and $w = 2$) yield no printed area.

18. We set $f(x) = x^3 - x - 1$ and note that $f(1) = -1$ while

$$f'(x) = 3x^2 - 1$$

and so $f'(1) = 2$. It follows that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{2} = \frac{3}{2}.$$

19. Recall that, using the limit of the right-endpoint approximations,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$. It looks like that if we set $f(x) = \sec^2 x$, $a = 0$, and $\Delta x = \frac{\pi}{4n}$ (which implies that $b - a = \frac{\pi}{4}$, or $b = \frac{\pi}{4}$), then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sec^2\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n} = \int_0^{\pi/4} \sec^2 x dx.$$

20. If we set

$$F(x) = \int_0^x \cos(u^2)du,$$

then the FTC gives that $F'(x) = \cos(x^2)$. Now $f(x) = F(5x)$, and so by the chain rule

$$f'(x) = 5F'(5x) = 5\cos(25x^2).$$

21. Set $u = x^2$, in which case $du = 2xdx$. Now when $x = 0$, $u = 0$; while when $x = \sqrt{\pi}$, $u = \pi$. It follows by substitution

$$\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2}\int_0^{\pi} \sin u du = \frac{1}{2}(-\cos u|_0^{\pi} = 1.$$

22. The curve $y = x^2 - 4x = x(x-4)$ has x-intercepts at $x = 0$ and $x = 4$ and is concave up. It intersects the curve $y = 2x$ at $x = 0$ and $x = 6$. It follows that the curve $y = 2x$ lies above the curve $y = x^2 - 4x$ between $x = 0$ and $x = 6$. Therefore, the area between the curves is given by

$$A = \int_0^6 \left(2x - (x^2 - 4x)\right) dx.$$

23. The curve $y = x - x^2$ is concave down and has x-intercepts $x = 0$ and $x = 1$. Hence the region in question is bounded by the curve $y = x - x^2$ above, $y = 0$ below, and extends between $x = 0$ and $x = 1$. We use the method of cylindrical shells. When we rotate around the line $x = 7$, each cylindrical shell has height $x - x^2$, thickness dx, and radius $7 - x$. So the volume of the solid is

$$\int_0^1 2\pi(7-x)(x-x^2)dx = 2\pi\int_0^1 (7-x)(x-x^2)dx.$$

24. Note that the intersection of the curves occurs at when $2 = 2 + 2x - x^2$, or $2x - x^2 = 0$...at $x = 0$ and $x = 2$. Since the curve $y = 2 + 2x - x^2$ is concave down, it follows that the curve $y = 2 + 2x - x^2$ lies above the curve $y = 2$ between $x = 0$ and $x = 2$. It follows that the solid indicated has cross-sections perpendicular to the x-axis which are annuli, with outer radius $2 + 2x - x^2$ and inner radius 2. Hence the volume is

$$\int_0^2 \pi\left((2 + 2x - x^2)^2 - 4\right) dx.$$

25. The average value of a function f on the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Hence we are looking for the value of the integral

$$\frac{1}{8}\int_0^8 \sqrt{16-2x}dx.$$

We make the substitution $u = 16 - 2x$, in which case $du = -2dx$ and so

$$\frac{1}{8}\int_0^8 \sqrt{16-2x}dx = -\frac{1}{16}\int_{16}^0 u^{1/2}du = -\frac{1}{16}\left(\frac{2}{3}u^{3/2}\right|_{16}^{0} = \left(\frac{1}{16}\right)\left(\frac{2}{3}\right)(16)^{3/2} = \frac{8}{3}.$$