# MATH 10550, EXAM 2 SOLUTIONS

(1) Find an equation for the tangent line to

$$x^2 + 2xy - y^2 + x = 2$$

at the point $(1, 2)$.

**Solution:** The equation of a line requires a point and a slope. The problem gives us the point so we only need to find the slope. Use implicit differentiation to get

$$2x + 2y + 2xy' - 2yy' + 1 = 0$$

Solve for $y'$

$$y' = \frac{2x + 2y + 1}{2y - 2x}$$

And evaluate at $(x, y) = (1, 2)$:

$$y' = \frac{2 + 4 + 1}{4 - 2} = \frac{7}{2}$$

The line passing through $(1, 2)$ with slope $\frac{7}{2}$ has the equation $y = \frac{7}{2}(x - 1) + 2 = \frac{7}{2}x - \frac{3}{2}$.

(2) The mass of a rod of length 10 cm is given by $m(x) = x^2 + \sqrt{x^2 + 9} - 3$ grams. What is the linear density of the rod at $x = 4$ cm?

**Solution:** The linear density is given by the derivative of the mass function. Calculate

$$m'(x) = 2x + \frac{x}{\sqrt{x^2 + 9}}$$

So the linear density at $x = 4$ is $m'(4) = 8 + \frac{4}{\sqrt{16+9}} = 8 + \frac{4}{10} = \frac{44}{5}$ g/cm.

(3) A man starts walking north from point P at a rate of 4 miles per hour. At the same time, a woman starts jogging west from point P at a rate of 6 miles per hour. After 15 minutes, at what rate is the distance between them changing?

**Solution:** To keep things simple make point $P$ the origin. Let $(x, 0)$ and $(0, y)$ be the positions of the woman and man at time $t$. Then their distance at $t$ is

$$d = \sqrt{x^2 + y^2}.$$

Then

$$d' = \frac{xx' + yy'}{\sqrt{x^2 + y^2}}.$$

It is given that $x' = 6$ and $y' = 4$. At $t = \frac{1}{4}$,

$$x = 6 \cdot \frac{1}{4} = \frac{3}{2}, \quad y = 4 \cdot \frac{1}{4} = 1,$$

and

$$d' = \frac{\frac{3}{2} \cdot 6 + 1 \cdot 4}{\sqrt{(\frac{3}{2})^2 + 1^2}} = \frac{13}{\frac{\sqrt{13}}{2}} = 2\sqrt{13}.$$

(4) Use a linear approximation to estimate $\sqrt[3]{(8.06)^2}$.

**Solution:** Take the function to be $f(x) = \sqrt[3]{x^2} = x^{2/3}$. Since $f(8) = 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$, the point $(8, 4)$ will be a good place to base the estimate. The slope of the tangent line at $(8, 4)$ is given by the derivative $f'(x) = \frac{2}{3}x^{-1/3}$ at $x = 8$: $f'(8) = \frac{2}{3}8^{-1/3} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$. So the tangent line has the equation $y = \frac{1}{3}(x - 8) + 4$. Thus the linear approximation is $\frac{1}{3}(8.06 - 8) + 4 = \frac{1}{3}(0.06) + 4 = 4.02$.

(5) Let

$$f(x) = x^4 - 24x^2 + 5x + 3.$$

Find the intervals where $f$ is concave up.

**Solution:** $f$ is concave up at a point $x$ if $f''(x) > 0$. We need to find all such $x$. Differentiate once to get $f'(x) = 4x^3 - 48x + 5$. And again to get $f''(x) = 12x^2 - 48$. Now solve the inequality

$$\begin{aligned} f''(x) &> 0 \\ 12x^2 - 48 &> 0 \\ 12x^2 &> 48 \\ x^2 &> 4 \end{aligned}$$

The last line means either $x > 2$ or $x < -2$. So the answer is $(-\infty, -2) \cup (2, \infty)$.

(6) Evaluate the limit

$$\lim_{x\to\infty} \frac{2 - 3x^2}{5x^2 + 4x}$$

**Solution:** Factor out an $x^2$ from top and bottom to get

$$\lim_{x\to\infty} \frac{2 - 3x^2}{5x^2 + 4x} = \lim_{x\to\infty} \frac{\frac{2}{x^2} - 3}{5 + \frac{4}{x}} = -\frac{3}{5}$$

(7) Suppose $f$ is continuous on $[2, 5]$ and differentiable on $(2, 5)$. If $f(2) = 1$ and $f'(x) \leq 3$ for $2 \leq x \leq 5$. According to the Mean Value Theorem, how large can $f(5)$ possibly be?

**Solution:** The mean value theorem says

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

for some $c$ in $(2, 5)$. Using the information given in the statement

$$\frac{f(5) - f(2)}{5 - 2} = \frac{f(5) - 1}{3} = f'(c) \leq 3$$

From $\frac{1}{3}(f(5) - 1) \leq 3$ some algebra gives $f(5) \leq 10$.

(8) Consider the function

$$f(x) = \frac{x}{x^2 + 9}.$$

One of the following statements is true. Which one?

**Solution:** Since $f$ is a rational function factor out an $x^2$ on the top and bottom to get

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$$

so $f$ has a horizontal asymptote of $y = 0$. This rules out all but 2 possible answers. To distinguish between the remaining two we need to see if $x = 3$ is a global minimum or maximum. The derivative is

$$f'(x) = \frac{9 - x^2}{(x^2 + 9)^2}$$

which is defined everywhere since the denominator is never zero. It is zero where the numerator is zero: $0 = 9 - x^2 = (3+x)(3-x)$. Thus $x = -3$ and $x = 3$ are critical points. Evaluate $f(3) = \frac{3}{18}$ and $f(-3) = -\frac{3}{18}$. Next, note that $f'(x) < 0$ in $(-\infty, 3) \cup (3, \infty)$ and $f'(x) > 0$ in $(-3, 3)$. Hence $f(x)$ is decreasing in $(-\infty, -3)$, then increasing in $(-3, 3)$ and then decreasing again in $(3, \infty)$. Hence $f(x)$ has a local maximum at $x = 3$ with value $f(3) = \frac{1}{6}$. This local maximum at $x = 3$ is in fact a global maximum since $y = 0$ is a horizonal asymptote.

(9) Let

$$f(x) = \frac{x}{x+2}.$$

After verifying that $f$ satisfies the hypothesis of the Mean Value Theorem on the interval $[0, 2]$, find all numbers $c$ that satisfy the conclusion of the Mean Value Theorem.

**Solution:** $f$ is a rational function whose denominator is zero at $x = -2$. Since $-2$ is not in the interval $[0, 2]$, $f$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the conditions of the mean value theorem are satisfied. Calculate $f(0) = 0$ and $f(2) = \frac{1}{2}$. We need to find all points $c$ in $(0, 2)$ with

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{\frac{1}{2} - 0}{2} = \frac{1}{4}$$

Use the quotient rule to get

$$f'(x) = \frac{2}{(x+2)^2}$$

And solve for $f'(x) = \frac{1}{4}$.

$$\begin{aligned} \frac{1}{4} &= \frac{2}{(x+2)^2} \\ (x+2)^2 &= 8 \\ x + 2 &= \pm\sqrt{8} = \pm 2\sqrt{2} \\ x &= \pm 2\sqrt{2} - 2 \end{aligned}$$

Since $-2\sqrt{2} - 2 < 0$ this solution is not in the range (0,2). But $2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$ is since $\sqrt{2} - 1$ is between 0 and 1. So the answer is $c = 2\sqrt{2} - 2$.

(10) Consider the function

$$f(x) = \frac{x}{x-1}.$$

One of the following statements is true. Which one?

**Solution:** Factor an $x$ out of the top and bottom to get

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1.$$

Thus the line $y = 1$ is a horizontal asymptote to $f$. There is a vertical asymptote at $x = 1$ where the denominator is zero. These observations rule out all but 2 of the answers. To distinguish between the two remaining we need to know where $f$ is concave down. Find

$$\begin{aligned} f'(x) &= -\frac{1}{(x-1)^2} \\ f''(x) &= 2(x-1)^{-3} \end{aligned}$$

$f''(x)$ is negative when $x - 1$ is negative (since raising to the $-3$ power preserves the sign), and $x - 1$ is negative when $x < 1$. Thus $f$ is concave down on the interval $(-\infty, 1)$.

(11) The position of a particle is given by

$$s(t) = t^5 - \frac{20}{3}t^3 + 6, \qquad \text{for } t \geq 0.$$

(a) When is the particle moving to the right?
(b) What is the total distance traveled between $t = 0$ seconds and $t = 3$ seconds?

**Solution:** (a) The particle is moving to the right when its velocity is positive. Calculate the velocity

$$v(t) = s'(t) = 5t^4 - 20t^2.$$

Factor it to find the zeros.

$$5t^4 - 20t^2 = 5t^2(t^2 - 4) = 5t^2(t-2)(t+2)$$

which means $v(t)$ is zero at $t = -2$, $t = 0$, and $t = 2$. Since $t \geq 0$ ignore the zero at $t = -2$. By calculation $v(1) = 5 \cdot -1 \cdot 3 < 0$ and $v(3) = 5 \cdot 9 \cdot 1 \cdot 5 > 0$, and thus since $v$ is continuous we know $v$ is negative on $(0, 2)$ and positive on $(2, \infty)$. So the particle is moving to the right when $t > 2$.

(b) To find the total distance traveled we need to add up the distance traveled in each direction during the time interval $[0, 3]$. For $t$ in $[0, 2]$ the particle is going left, and on $[2, 3]$ it is going to the right.

$$\text{For } [0,2] \qquad |s(2) - s(0)| = |32 - \frac{20}{3} \cdot 8 + 6 - 6| = \frac{64}{3}$$

$$\text{For } [2,3] \qquad |s(3) - s(2)| = |3^3(9 - \frac{20}{3}) + 6 - 8(4 - \frac{20}{3}) - 6| = |27 \cdot \frac{7}{3} + \frac{64}{3}| = \frac{253}{3}$$

So the total distance travelled is $\frac{64}{3} + \frac{253}{3} = \frac{317}{3}$ units.

(12) A melting ice cube is decreasing in volume at a rate of 10 $\text{cm}^3$/minute, but remains a cube as it melts.

(a) How fast are the edges of the cube decreasing when the length of each edge is 20 cm?
(b) How fast is the surface area of the cube decreasing when the length of each edge is 20 cm?

**Solution:** (a) Given a side of length $x$, the volume of the cube is $V = x^3$. If we think of $V$ and $x$ as functions of time ($t$) then the derivative of $V$ with respect to $t$ is

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

The volume is decreasing at 10 $\text{cm}^3$/min., which means $\frac{dV}{dt} = -10$. So

$$\frac{dx}{dt} = \frac{-10}{3x^2}.$$

When $x = 20$ we have $\frac{dx}{dt} = \frac{-10}{3 \cdot 20^2} = -\frac{1}{120}$ cm/min.

(b) Given the side length $x$, the cube has six square faces so the total surface area is $A = 6x^2$. Think of $A$ and $x$ as functions of $t$ and take the derivative:

$$\frac{dA}{dt} = 12x \frac{dx}{dt}$$

When $x = 20$ we know $\frac{dx}{dt} = -\frac{1}{120}$ from part (a). Substuting in these values gives $\frac{dA}{dt} = 12 \cdot 20 \cdot \frac{-1}{120} = -\frac{240}{120} = -2\,\text{cm}^2/\text{min}$.

(13) Let

$$f(x) = 3x^4 + 16x^3 - 30x^2 - 2$$

(a) What are the critical numbers for $f$?
(b) If we restrict $f$ to the interval $[-1, 1]$, give the $x$ and $y$ values for the global maximum and the global minimum for $f$ on this interval.

**Solution:** (a) Critical numbers are values for $x$ where $f'(x)$ either doesn't exist or is equal to 0. The derivative is $f'(x) = 12x^3 + 48x^2 - 60x$ which is a polynomial, so it is defined everywhere. To find the zeros, factor $f'$

$$12x^3 + 48x^2 - 60x = 12x(x^2 + 4x - 5) = 12x(x + 5)(x - 1)$$

Thus $f'$ has zeros at $x = 0$, $x = -5$ and $x = 1$ and these are all the critical numbers for $f$.

(b) The only critical numbers in the interval $[-1, 1]$ are $x = 0$ and $x = 1$. Evaluate at these critical points and at the endpoints:

$$f(-1) = -45$$
$$f(0) = -2$$
$$f(1) = -13$$

So the maximum is at $(0, -2)$ and the minimum is at $(-1, -45)$.