Project 2: Taylor polynomials

The attached Project appears in Section 2.8 of your book.

To plot the graph of the function \( f(x) \), you may use any graphing software. **Please include graphs of the relevant functions and polynomials with your project.**

The proper syntax for Mathematica is given below:

To graph the function \( f(x) = 3x^2 + 2x + 1 \) on the interval \([-4, 4]\) with Mathematica, you can use the command :

\[
\text{In}[1] := \text{Plot}[3x^2 + 2x + 1, \{x, -4, 4\}]
\]

To execute this command, click on the command, then choose Evaluation from the main menu, and choose Evaluate Cells from the menu. Alternatively click on the command and Press Shift Return on your keyboard.

To graph the functions \( y = 3x^2 + 2x + 1 \), \( y = x + 4 \) and \( y = x^3 \) simultaneously on the interval \([-4, 4]\), you can use the Plot command as follows:

\[
\text{In}[2] := \text{Plot}[\{3x^2 + 2x + 1, x + 4, x^3\}, \{x, -4, 4\}]
\]

You may download Mathematica for free from the OIT website under Software. It is also available on all university computers in the computer labs.
The linear approximation to $f(x)$ at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0) = 1 + 0 \cdot x = 1.$$ 

Let $P(x) = A + Bx + Cx^2$ be a quadratic approximation to $f(x) = \cos x$ with

1. $P(0) = f(0)$
2. $P'(0) = f'(0)$
3. $P''(0) = f''(0)$

We have $P(0) = A + B \cdot 0 + C \cdot 0^2 = A$.
We have $P'(x) = B + 2Cx$ and $P'(0) = B + 2C \cdot 0 = B$.
We have $P''(0) = 2C$.

Now $f(0) = \cos 0 = 1$, $f'(0) = -\sin 0 = 0$ and $f''(0) = -\cos 0 = -1$. Therefore our three conditions amount to

1. $P(0) = f(0)$ or $A = 1$.
2. $P'(0) = f'(0)$ or $B = 0$.
3. $P''(0) = f''(0)$ or $2C = -1$.

From this we see that the values of $A$, $B$ and $C$ are completely determined by the derivatives of $f(x)$ at 0.

$$A = f(0) = 1, \quad B = f'(0) = 0, \quad C = \frac{f''(0)}{2} = -\frac{1}{2}.$$

Thus we see that

$$P(x) = 1 - \frac{x^2}{2}.$$

The graphs of $f(x) = \cos x$, its linearization at $x = 0$, $L(x) = 1$ and its quadratic approximation $P(x) = 1 - \frac{x^2}{2}$ are shown below.
2. Determine the values of $x$ for which the quadratic approximation $f(x) = P(x)$ in Problem 1 is accurate to within 0.1. [Hint: Graph $y = P(x)$, $y = \cos x - 0.1$, and $y = \cos x + 0.1$ on a common screen.]

Below we show the plot of the three curves $y = \cos x + 0.1$, $y = \cos x - 0.1$ and $y = P(x) = 1 - \frac{x^2}{2}$. We see that the graph of $y = P(x)$ stays within the bounds determined by the curves $y = \cos x + 0.1$ and $y = \cos x - 0.1$ on the interval $-1.25 < x < 1.25$. This means that $|P(x) - \cos x| < 0.1$ for $-1.25 < x < 1.25$ of the quadratic approximation to $\cos x$ at $x = 0$ is accurate up to an error of $\pm 0.1$ when $-1.25 < x < 1.25$.

3. To approximate a function $f$ by a quadratic function $P$ near a number $a$, it is best to write $P$ in the form

$$P(x) = A + B(x - a) + C(x - a)^2$$

Show that the quadratic function that satisfies conditions (i), (ii), and (iii) is

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$$

Let $P(x)$ be a quadratic approximation to a function $f(x)$ at $x = a$ of the form

$$P(x) = A + B(x - a) + C(x - a)^2.$$ We would like the first and second derivatives of $P(x)$ to match those of $f(x)$ at $x = a$ and we would also like $P(a) = f(a)$. Therefore we must find such a quadratic with

1. $P(a) = f(a)$
2. $P'(a) = f'(a)$
3. \( P''(a) = f''(a) \)

\[
P(a) = A + B \cdot 0 + C \cdot 0 = A.
\]

\[
P'(x) = 0 + B + C \cdot 2(x - a)^1 \cdot 1 = B + 2C(x - a), \text{ therefore } P'(a) = B.
\]

\[
P''(x) = 2C, \text{ therefore } P''(a) = 2C.
\]

Therefore conditions 1-3 above can be restated as

1. \( A = f(a) \)

2. \( B = f'(a) \)

3. \( 2C = f''(a) \)

This means that the values of \( A, B \) and \( C \) for such a polynomial are completely determined and the **quadratic approximation to** \( f(x) \) at \( x = a \) is given by

\[
P(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2.
\]

4. Find the quadratic approximation to \( f(x) = \sqrt{x} + 3 \) near \( a = 1 \). Graph \( f \), the quadratic approximation, and the linear approximation from Example 2 in Section 2.8 on a common screen. What do you conclude?

The quadratic approximation to \( f(x) = \sqrt{x} + 3 \) at \( a = 1 \) is given by

\[
P(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2.
\]

\[
f(1) = \sqrt{1 + 3} = \sqrt{4} = 2.
\]

\[
f'(x) = \frac{1}{2}(x + 3)^{-1/2}, \text{ therefore } f'(1) = \frac{1}{2}(1 + 3)^{-1/2} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.
\]

\[
f''(x) = \frac{1}{2} \cdot -\frac{1}{2}(x + 3)^{-3/2}, \text{ therefore } f''(1) = \frac{1}{2} \cdot -\frac{1}{2}(4)^{-3/2} = \frac{1}{2} \cdot -\frac{1}{2} \cdot \frac{1}{8} = -\frac{1}{32}.
\]

Therefore the quadratic approximation to \( f(x) = \sqrt{x} + 3 \) at \( a = 1 \) is given by

\[
P(x) = 2 + \frac{1}{4}(x - 1) + \frac{-1}{64}(x - 1)^2.
\]

The linear approximation to \( f(x) = \sqrt{x} + 3 \) at \( a = 1 \) is given by

\[
L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1)
\]

We show the graphs of \( f(x), L(x) \) and \( P(x) \) near \( x = 1 \) below.
As you can see the quadratic approximation gives a better approximation to the values of the functions $f(x) = \sqrt{x} + 3$ for values of $x$ near $x = 1$.

5. Instead of being satisfied with a linear or quadratic approximation to $f(x)$ near $x = a$, let’s try to find better approximations with higher-degree polynomials. We look for an $n$th-degree polynomial

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$$

such that $T_n$ and its first $n$ derivatives have the same values at $x = a$ as $f$ and its first $n$ derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2} f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the $n$th-degree Taylor polynomial of $f$ centered at $a$.

Suppose that we have a polynomial of degree $n$,

$$P(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n,$$

for which $f(a) = P(a)$, $f'(a) = P'(a)$, $f''(a) = P''(a)$, $\ldots$, $f^{(n)}(a) = P^{(n)}(a)$.

Then we have $P(a) = c_0$.

$P'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$ and $P'(a) = c_1$.

$P''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + \cdots + n \cdot (n-1) \cdot c_n(x-a)^{n-2}$ and $P''(a) = 2c_2$.

$P^{(3)}(x) = 3 \cdot 2 \cdot 1c_3 + \cdots + n \cdot (n-1) \cdot (n-2) \cdot c_n(x-a)^{n-3}$ and $P^{(3)}(a) = 3 \cdot 2 \cdot 1c_3$.

$\vdots$

$P^{(n)}(x) = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 1 \cdot c_n = n!c_n = P^{(n)}(a)$.

Now equating $P^{(k)}(a)$ with $f^{(k)}(a)$ for $1 \leq k \leq n$, we get

$$f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2 \quad \text{or} \quad c_2 = \frac{f''(a)}{2}, \quad f^{(3)}(a) = 3 \cdot 2 \cdot 1 \cdot c_3 \quad \text{or} \quad c_3 = \frac{f^{(3)}(a)}{3 \cdot 2 \cdot 1}$$

$\ldots f^{(k)}(a) = k!c_k \quad \text{or} \quad c_k = \frac{f^{(k)}(a)}{k!}$ \quad $\ldots$, $f^{(n)}(a) = n!c_n \quad \text{or} \quad c_n = \frac{f^{(n)}(a)}{n!}$

This gives us that

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
6. Find the 8th-degree Taylor polynomial centered at \( a = 0 \) for the function \( f(x) = \cos x \). Graph \( f \) together with the Taylor polynomials \( T_2, T_4, T_6, T_8 \) in the viewing rectangle \([-5, 5]\) by \([-1.4, 1.4]\) and comment on how well they approximate \( f \).

The 8th degree Taylor polynomial of \( f(x) = \cos x \) at \( x = 0 \) is given by

\[
T_8(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(8)}(0)}{8!}x^8.
\]

\( f(x) = \cos x, \quad f(0) = 1 \)
\( f'(x) = -\sin x, \quad f'(0) = 0 \)
\( f''(x) = -\cos x, \quad f''(0) = -1 \)
\( f'''(x) = \sin x, \quad f'''(0) = 0 \)
\( f^{(4)}(x) = \cos x, \quad f^{(4)}(0) = 1 \)
\( f^{(5)}(x) = -\sin x, \quad f^{(5)}(0) = 0 \)
\( f^{(6)}(x) = -\cos x, \quad f^{(6)}(0) = -1 \)
\( f^{(7)}(x) = \sin x, \quad f^{(7)}(0) = 0 \)
\( f^{(8)}(x) = \cos x, \quad f^{(8)}(0) = 1 \)

Therefore

\[
T_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.
\]

In fact, we see that

\[
T_2(x) = 1 - \frac{x^2}{2!},
\]
\[
T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},
\]
\[
T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.
\]

We show the graph all four Taylor polynomials with the function \( f(x) = \cos x \) below.

In the graph below, we show the graph of \( \cos x \) along with \( T_8(x) \), showing that it gives the best polynomial approximation among those shown above.