Project 4: Introduction to Circuits

The attached Project was prepared by Professor Yih-Fang Huang from the department of Electrical Engineering. The examples given are example of basic problems that you will encounter in a circuits class. This worksheet is recommended for students planning to take a circuits class in Engineering.

Please read the introduction and read through the examples given paying particular attention to how to set up the system of linear equations involved. Then complete the work involved in solving the systems of linear equations for four out of six of the examples given. Please show all of your work for the solution of each system of linear equations.

Cramer's Rule

You can find information on Cramer's rule on the internet, for example:

http://mathworld.wolfram.com/CramersRule.html

You will also find information on calculating determinants of matrices:

http://mathworld.wolfram.com/Determinant.html

Examples for Introduction to Electrical Engineering

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1   Basic Terminologies

An *electric circuit* is a collection of interconnected multi-terminal electrical devices (also referred to as circuit elements). Conventional circuit analysis focuses on devices that can be modeled as *two-terminal devices*, namely, a circuit element with two leads coming out of it. Examples of two-terminal devices include independent sources (e.g., a battery), simple resistors, capacitors, inductors and diodes. In practice, there are also many devices that have more than two terminals. Typical such examples are transistors, transformers, and operational amplifiers. In circuit analysis, those multi-terminal devices are typically modeled as two-terminal devices with dependent sources.

Before we proceed to show some circuit analysis examples, we shall define some terminologies first:

- **Node**: a node is a point of connection between two or more circuit elements.

- **Branch**: a branch is a portion of the circuit that consists of *one* circuit element and its two terminal nodes.

- **Loop**: a (closed) loop is a sequence of connected branches that begin and end at the same node.

- **Branch Current**: a branch current is the current that flows through the electrical device of the branch. The physical unit of current is *ampere*, named after the French mathematician and physicist, André-Marie Ampère (1775-1836).

- **Branch Voltage**: a branch voltage is the potential difference between the two terminal nodes of the electrical device of the branch. The physical unit of voltage is *volt*, which is named in honor of the Italian physicist Alessandro Volta (1745-1827), who invented the voltaic pile, possibly the first chemical battery.

- **Node Voltage**: a node voltage is the potential difference between a designated node and the reference node.

- **Loop Current**: a loop current is the current that flows through a closed loop. Note that loop current is different from branch current if that branch is common to two or more loops.

It is important to note that conventional circuit analysis assumes that all wires (or lead) are perfect conductors, namely, the *lumped parameter circuit* model. Therefore, no energy is lost when currents flow through the wires, and non-zero branch voltages occur only across a circuit element (a wire itself is not a circuit element).

In circuit analysis, each branch has two branch variables, i.e., branch voltage and branch current. These are *algebraic variables*, thus they have signs associated with their values. A circuit is completely characterized if all the branch variables are known. Thus the objective of circuit analysis is to solve for all branch voltages and branch currents.

2   Circuit Analysis

In principle, three sets of equations are sufficient to perform circuit analysis for any circuit, which are *Kirchoff's current law* (KCL), *Kirchoff's voltage law* (KVL), and branch element equations (e.g., *Ohm's*

law). The KCL and KVL together characterize the circuit topology, i.e., how the circuit elements are connected, while the branch element equation characterizes the physical property, specifically, the current-voltage relation, of the circuit element.

2.1 Kirchhoff Laws

The Kirchhoff Laws are results of conservation principles that must always be obeyed by any electric circuit. These laws are named after Gustav Robert Kirchhoff (1824-1887), a German physicist who contributed to the fundamental understanding of electrical circuits and who formulated these laws in 1845 when he was still a student at University of Königsberg in Germany. Simply stated, KCL is concerned with the currents entering a node, while KVL is stated with respect to the branch voltages around a loop in a circuit. Note that these Kirchhoff laws hold regardless of the physical properties of the circuit elements.

KCL: *The algebraic sum of all currents entering any node is zero.*

KVL: *The algebraic sum of the branch voltages around any closed loop equals zero.*

KCL is simply a statement that charges cannot accumulate at the nodes of a circuit. This principle is identical to concepts found in fluid dynamics. Namely that if you look at the fluid flowing into one end of a pipe, you expect the same amount of fluid to flow out the other end. If this did not occur, then fluid would accumulate in the pipe and eventually cause the pipe to burst. KCL is nothing more than an electrical equivalent of this intuitive physical idea from fluid mechanics. In a nutshell, KCL is a charge conservation law, while KVL is an energy conservation law which states, in essence, that the total work done in going around a loop will be zero.

To employ KVL and KCL, we must always keep in mind that, in circuit analysis, the currents and voltages are *algebraic variables*. As such, it is helpful to have a sign convention for those variables. To facilitate the subsequent discussion, we adopt the following convention: (1) when applying KCL to a node, we designate any current flowing into the node as positive and any leaving the node as negative; (2) when applying KVL around a closed loop, we travel around the loop clockwise, and a branch voltage is positive if we enter the positive polarity of the voltage first, otherwise it is negative.

To explain how KCL works, let us consider the circuit shown in Figure 1. This figure shows a circuit consisting of three branches, an independent voltage source connected in parallel with two resistors. This is also commonly known as a *single-node* circuit, for there is only one node a, apart from the ground node b. The single node a of this circuit is shown in the righthand drawing of Figure 1. At this node, we see three currents. Two of these currents i_1 and i_2 are leaving node a and the third current i_0 is entering node a. By the sign convention discussed above and by KCL, we can write an equation as below:

$$i_0 - i_1 - i_2 = 0$$

To illustrate applications of KVL, consider the single-loop circuit shown in Figure 2. This is the loop formed from branches

$$(a, b) \rightarrow (b, c) \rightarrow (c, a)$$

The voltages obtained by traversing this loop are

$$v_{ab}, v_{bc}, V$$

In this figure, we start at node a and begin tracing out our loop in a clockwise direction, we see that the traverse of branch (a, b) goes from $+$ to $-$. This is considered as a negative change in potential (i.e. we
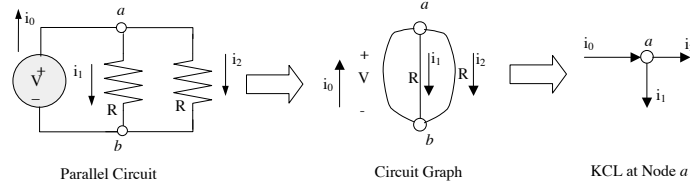
Figure 1: KCL at node b

are decreasing the potential). The same is true for the voltage over branch (b, c). Note, however, that in traversing branch (c, a) that we are going from a negative to positive polarity. The change in potential, therefore, is positive. On the basis of our preceding discussion, we can see that KVL will lead to the following equation:
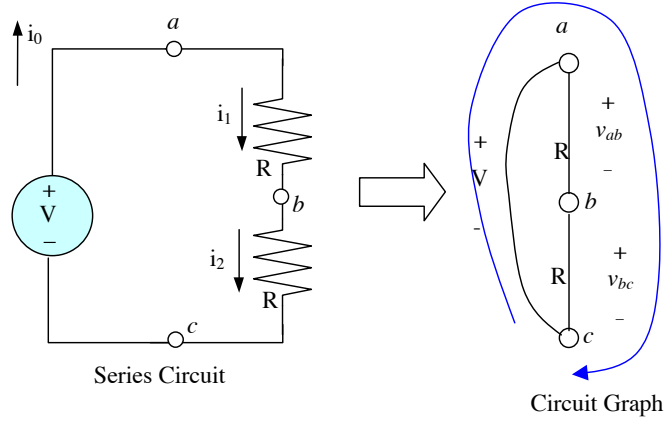
$$-V + v_{ab} + v_{bc} = 0$$

Figure 2: A Simple Resistive Circuit

2.2 Branch Element Equations

The third set of equations needed in performing circuit analysis is the set of *branch element equations*, which are equations that characterize the relation between the branch voltage and branch current (i.e., the V-I relation) and that relation is due to the physical properties of the electrical device of the branch. In our discussion, and in most two-terminal device modeling, the following three equations are most commonly used:

1. For an ideal resistor, the *Ohm's Law* states that the branch voltage is directly proportional to the branch current, namely,

$$v(t) = Ri(t) \tag{1}$$

   where the proportionality R is called the resistance and is measured in *Ohms*, denoted by a Greek symbol Ω, named after the German physicist Georg Simon Ohm (1789-1854). Equation (1) is the well-known *Ohm's Law*.

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2. For an ideal capacitor, the branch current is directly proportional to the derivative of the branch voltage, namely,

$$i(t) = C\frac{dv(t)}{dt} \tag{2}$$

where the proportionality constant C is called the capacitance and is measured in *farads* named in honor of the English Chemist and Physicist Michael Faraday (1791-1867).

3. For an ideal inductor, the branch voltage is directly proportional to the derivative of the branch current, namely,

$$v(t) = L\frac{di(t)}{dt} \tag{3}$$

where the proportionality constant L is called the inductance and is measured in *henrys*, named after an American scientist Joseph Henry (1797-1878).

2.3   Nodal Analysis and Mesh Analysis

We have learned in previous sections that three sets of equations (KCL, KVL, and Ohm's law) are sufficient to perform circuit analysis for any circuits. To be more specific, in solving a circuit that comprises B branches (i.e., it has B circuit elements) and N nodes, we are solving for $2B$ variables, i.e., B branch voltages and B branch currents. Accordingly, we need $2B$ independent equations to obtain unambiguous solutions. For a circuit with B branches and N nodes, there will be $N-1$ independent KCL equations, $B-N+1$ independent KVL equations, and B branch element equations. Thus there are, altogether, $2B$ independent equations. However, since B is usually large (much larger than N), solving $2B$ equations may be an onerous task. The simpler alternatives are *nodal analysis* and *mesh analysis*. Both methods are derived from the aforementioned three sets of equations, however, they solve a smaller set of alternative algebraic variables. The nodal analysis first defines node voltages for each node as the potential difference between the designated node and the reference (ground) node. Then, it employs KCL to set up the so-called node equations, expressing every term in the equation in terms of node voltages. Thus each node equation is a KCL equation, except that the unknown algebraic variables are node voltages. Once all node voltages are solved (with the node equations), we can obtain all branch voltages and branch currents. In essence, instead of directly solving for all branch voltages and branch currents, Nodal Analysis solves for node voltages and then use those solutions to determine all branch voltages and branch currents. Similarly, the mesh analysis defines mesh currents for each mesh (a closed loop) as the current traversing around this closed loop. Then, it employs KVL to set up the so-called mesh equations, expressing wherever possible every term in the equation in terms of mesh currents. Thus each mesh equation is a KVL equation, except that the unknown algebraic variables are mesh currents. Once all mesh currents are solved (with the mesh equations), we can obtain all branch voltages and branch currents. Since the number of nodes (or that of meshes) is significantly less than the number of branches, those two methods involve many fewer variables to solve for. All circuits can be solved either by nodal analysis or by mesh analysis.

To facilitate the employment of nodal analysis and mesh analysis, we need to introduce another sign convention, i.e., the so-called *associated passive sign convention* which is defined in accordance with the physical properties of *passive* circuit elements, i.e., circuit elements like resistors that absorb energy. The *associated passive sign convention* states that the branch current of a passive circuit element flows from high to low potential. Accordingly, the sign of the branch current is positive if it flows from the positive (+) polarity of the voltage to the negative (-) polarity. As such, unless otherwise specified, once the polarity of the voltage is defined, the direction of associated branch current is defined. The converse is also true. In other words, if the current flow direction is marked, then the voltage polarity is defined accordingly.

The following section presents some examples that use nodal analysis and mesh analysis for simple resistive circuits. The mathematical tools involved are simply that of systems of linear equations. Those equations can be solved by hand (if the number of unknowns is less than or equal to 3), using Cramer's rule, or using a computer program like Matlab.

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3 Examples for Linear Resistive Circuit Analysis
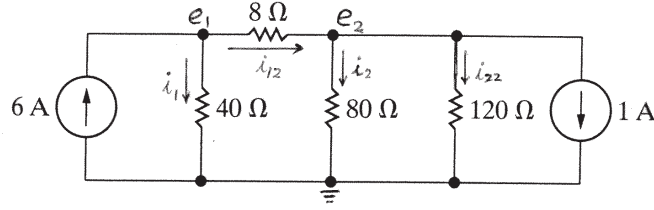
Example 1 (Nodal Analysis)

Figure 3: Example 1

In this example, we employ nodal analysis to find the branch voltages and branch currents in the circuit shown. We begin by marking a reference node as shown. Other than the reference node, there are only two nodes, namely, Node 1 and Node 2. We thus define two node voltages, e_1 and e_2. Note that e_1 is defined such that its positive polarity is at node 1 and negative polarity is at the reference node. This is the convention that we use in defining all node voltages. Since there are only two non-reference nodes, we only need to set up two node equations to solve for those node voltages. The node equations are simply KCL equations. Three branches are connected to Node 1 - an independent current source of 6 Amps, a 40Ω resistor and an 8Ω resistor. Accordingly, there are three branch currents in the KCL equation, which states that the algebraic sum of those three branch currents is equal to zero, namely,

$$6 - i_1 - i_{12} = 0 \tag{4}$$

We can see that four branches are connected to Node 2 - three resistor branches and a 1 Amp independent current source. Thus the KCL equation for Node 2 is,

$$i_{12} - i_2 - i_{22} - 1 = 0 \tag{5}$$

In writing the above equations, we have adopted the convention that currents entering a node is positive while currents leaving a node is negative. Now, we need to convert the above equations into *node equations*, expressing all unknown currents as functions of the node voltages by using branch element equations (Ohm's Law in this case, since there are only resistors) and KVL equations wherever needed. In particular,

$$
\begin{aligned}
i_1 &= e_1/40 \\
i_2 &= e_2/80 \\
i_{22} &= e_2/120
\end{aligned}
$$

In addition to the above expressions, we need to express i_{12} in terms of the node voltages. Since i_{12} is marked as flowing from left to right on the 8Ω resistor branch, the associated branch voltage must be defined to have positive polarity at Node 1 and negative polarity at Node 2, according to the *associated passive sign convention* described in Section 2.3. As such, the branch voltage is $e_1 - e_2$, which can be seen easily by applying KVL around the closed loop that encompasses the three branches - 40Ω, 8Ω, and 80Ω. Hence $i_{12} = (e_1 - e_2)/8$. Now, we are ready to write down the two node equations. Substituting all the current terms expressed in node voltages into Eqs. (4, 5) yields

$$6 - \frac{e_1}{40} - \frac{(e_1 - e_2)}{8} = 0 \tag{6}$$

$$\frac{(e_1 - e_2)}{8} - \frac{e_2}{80} - \frac{e_2}{120} - 1 = 0 \tag{7}$$

We now have just two algebraic variables, e_1 and e_2 to solve for, with two independent equations. Solving those two equations, (6), (7), yields $e_1 = 120$ Volts and $e_2 = 96$ Volts. Now, you can verify that once e_1 and e_2 are known, we can find all the branch voltages and branch currents.
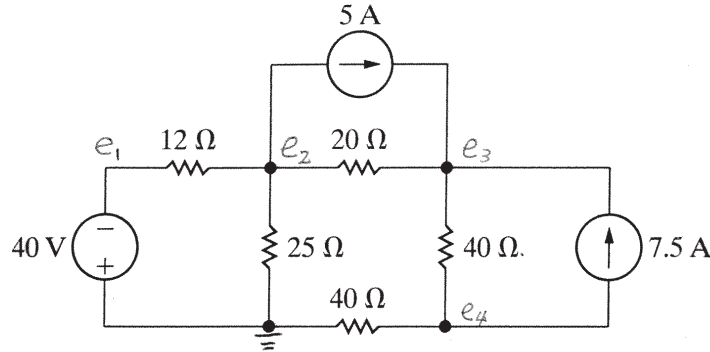
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Example 2 (Nodal Analysis)

Figure 4: Example 2

In this example, we can see that there are four nodes in addition to the reference node, thus there are four node voltages to solve for. Note that this reference node has been chosen intentionally so that one of the node voltages is given, -40 Volts, negative of the independent voltage source, thereby reducing by one the number of variables to solve for. To solve for the circuit shown in Figure 4, we proceed similarly as in Example 1, except now e_1 is already given. Thus the node equations are as follows:

$$
\begin{aligned}
\text{At Node 1:} \quad & e_1 & = & \quad -40 \\[2mm]
\text{At Node 2:} \quad & \frac{(e_1-e_2)}{12} - \frac{e_2}{25} - 5 - \frac{(e_2-e_3)}{20} & = & \quad 0 \\[2mm]
\text{At Node 3:} \quad & 5 + \frac{(e_2-e_3)}{20} - \frac{(e_3-e_4)}{40} + 7.5 & = & \quad 0 \\[2mm]
\text{At Node 4:} \quad & \frac{(e_3-e_4)}{40} - 7.5 - \frac{e_4}{40} & = & \quad 0
\end{aligned}
\tag{8}
$$

Now, we can solve for e_2, e_3, and e_4 from the equations in (8) - three variables with three independent equations. The solutions are $e_2 = -10$ Volts, $e_3 = 132$ Volts, and $e_4 = -84$ Volts. Again, you should verify that once you know all the node voltages, you can easily determine all branch voltages and branch currents by applying Ohm's Law and KVL. Before we leave this example, you may notice that the solution process may become more complicated if we assign the reference node differently. More explicitly, you will need to solve for four unknown variables with four equations, instead of three variables with three equations, as we shall see in the next example.
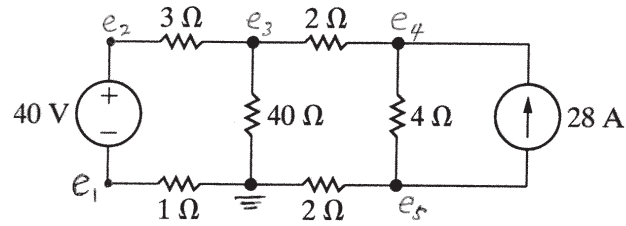
Example 3 (Nodal Analysis)

Figure 5: Example 3

There are five nodes, in addition to the reference node, in the circuit shown. The node equations are:

Combining Node 1 and Node 2: $\dfrac{-e_1}{1} - \dfrac{(e_2 - e_3)}{3}$ $= \quad 0$

At Node 3: $\dfrac{(e_2 - e_3)}{3} - \dfrac{e_3}{40} - \dfrac{(e_3 - e_4)}{2}$ $= \quad 0$

At Node 4: $\dfrac{(e_3 - e_4)}{2} - \dfrac{(e_4 - e_5)}{4} + 28$ $= \quad 0$ (9)

At Node 5: $\dfrac{(e_4 - e_5)}{4} - \dfrac{e_5}{2} - 28$ $= \quad 0$

Additional equation: $e_2 - e_1$ $= \quad 40$

Solving for five variables with the five equations in (9) yields $e_1 = 5$ Volts, $e_2 = 45$ Volts, $e_3 = 60$ Volts, $e_4 = 73$ Volts, and $e_5 = -13$ Volts. This problem can also be solved with a Matlab program. In this example, the direct application of Nodal analysis calls for solution of five unknown variables. However, further investigation may lead you to see that you can simplify this problem to one of solving three variables with three equations by exploring the first and the last equations in (9) to eliminate e_1 and e_2.
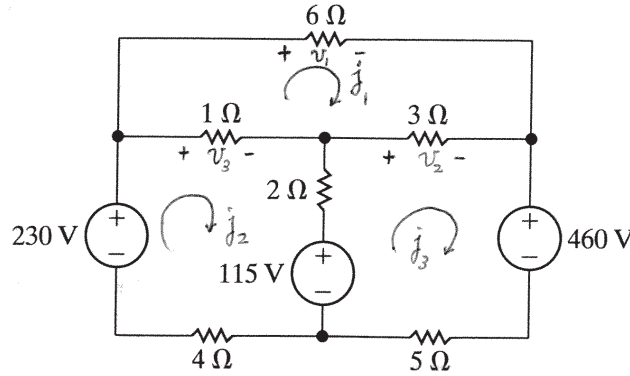
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Example 4 (Mesh Analysis)

Figure 6: Example 4

In mesh analysis, our goal is to solve for the mesh currents using mesh equations. The mesh equations are KVL equations with the unknown branch voltages expressed in terms of mesh currents. In this example, there are three meshes (also termed fundamental loops). Employing KVL equation in Mesh 1 yields,

$$v_1 - v_2 - v_3 = 0 \tag{10}$$

In writing (10), we have adopted the convention of transversing the loop clockwise, and if the polarity of the branch voltage results in a branch current defined through the *associated passive sign convention* flowing in the same direction as the mesh current direction when it transverses through the branch, that branch voltage is a positive variable. Now we need to convert (10) into a mesh equation. This is accomplished by employing branch element equation (Ohm's Law in this case, since there are only resistors). Note that the branch current through the 6Ω resistor branch is j_1, the branch current through the 3Ω resistor branch is $-j_1 + j_3$, and the branch current through the 1Ω resistor branch is $-j_1 + j_2$. Again, we have adopted *associated passive sign convention* in defining those branch currents, following the polarity of the branch voltages that had been marked. Applying Ohm's Law, we have:

$$
\begin{aligned}
v_1 &= j_1 6 \\
v_2 &= (-j_1 + j_3)3 \\
v_3 &= (-j_1 + j_2)1
\end{aligned}
$$

Substituting the above expressions into (10) yields,

$$j_1 6 - (-j_1 + j_3)3 - (-j_1 + j_2) = 0 \tag{11}$$

By the same token, we can derive the mesh equations for Mesh 2 and Mesh 3, noting that the branches with independent voltage sources have known branch voltages equal to the value of the voltage source. Hence the three mesh equations are:

$$
\begin{aligned}
\text{Mesh 1:} \quad & j_1 6 - (-j_1 + j_3)3 - (-j_1 + j_2) = 0 \\
\text{Mesh 2:} \quad & -230 + (-j_1 + j_2) + (j_2 - j_3)2 + 115 + j_2 4 \;=\; 0 \\
\text{Mesh 3:} \quad & -115 - (j_2 - j_3)2 + (j_3 - j_1)3 + 460 + j_3 5 \;=\; 0
\end{aligned}
\tag{12}
$$

Here we have three independent equations to solve for three unknown variables, j_1, j_2, and j_3. The solutions are $j_1 = -10.6$ Amps, $j_2 = 4.4$ Amps, and $j_3 = -36.8$ Amps. Once again, you may verify that once we know the values of all mesh currents, we can determine all branch voltages and branch currents.

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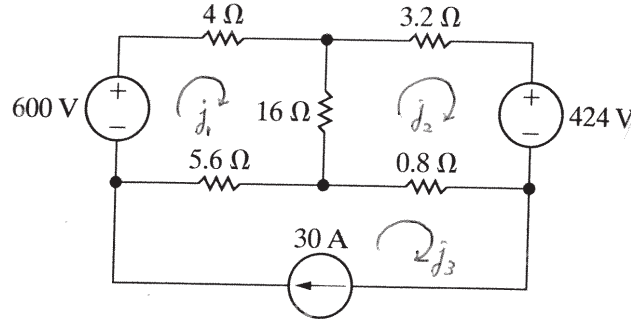**Example 5** (Mesh Analysis)

Figure 7: Example 5

This example demonstrates that when there is an independent current source in an isolated branch, i.e., a branch that is not common to two or more meshes, we can reduce by one the number of mesh currents to be solved for. For the circuit shown in Figure 7, we see that $j_3 = 30$ Amps. Thus we only need to solve for j_1 and j_2. Following the same procedures as in Example 4, we can write the mesh equations.

$$
\begin{aligned}
\text{Mesh 1:} & \quad -600 + j_1 4 + (j_1 - j_2)16 + (j_1 - j_3)5.6 & = & \quad 0 \\
\text{Mesh 2:} & \quad -(j_1 - j_2)16 + j_2 3.2 + (j_2 - j_3)0.8 + 424 & = & \quad 0
\end{aligned}
\tag{13}
$$

Solving for j_1 and j_2 with the equations in (13) yields $j_1 = 35$ Amps and $j_2 = 8$ Amps.
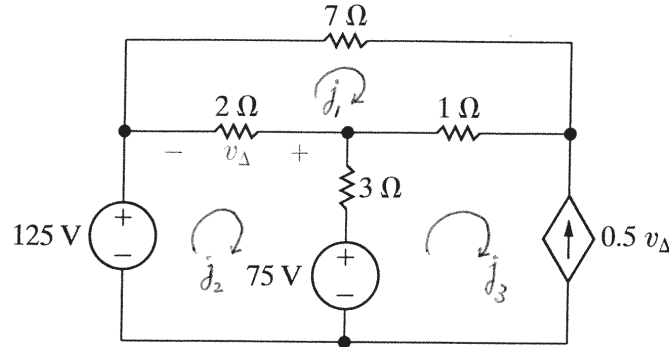
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Example 6 (Mesh Analysis)

Figure 8: Example 6

In this example, we show that when the circuit contains dependent sources[1], we may need to add more equations for solve for all the unknown variables. In principle, we need the same number of independent equations as the number of unknown variables to obtain the unique solution, as we have learned in Linear Algebra. The mesh equations for the circuit shown in Figure 8 are:

$$
\begin{aligned}
\text{Mesh 1:} \quad & j_1 7 + (j_1 - j_3)1 + (j_1 - j_2)2 && = && 0 \\
\text{Mesh 2:} \quad & -125 - (j_1 - j_2)2 + (j_2 - j_3)3 + 75 && = && 0 \\
\text{Mesh 3:} \quad & j_3 && = && -0.5 v_\Delta \\
\text{Additional equation:} \quad & v_\Delta = (j_1 - j_2)2
\end{aligned}
\tag{14}
$$

In summary, there are four equations for solving four unknown variables, j_1, j_2, j_3, and v_Δ. The solutions are $j_1 = 6$ Amps, $j_2 = 22$ Amps, $j_3 = 16$ Amps, and $v_\Delta = -32$ Volts.

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[1]There are basically four types of dependent sources that we may encounter in those simple resistive circuits: voltage-controlled voltage source, current-controlled voltage source, voltage-controlled current source and current-controlled current source

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