Antiderivatives

Sometimes we are given a function $f(x)$, and wish to find a function for which $f(x)$ is the derivative, that is, we want to find a function $F(x)$ with $F'(x) = f(x)$. For example if I knew the velocity of a particle over a time interval, I may also like to know its position function on that time interval. If we know the rate at which oil is leaking from an oil well, we may wish to use it to find out how much oil had leaked from the well at any given time.

Definition A function F is called an antiderivative for the function f on an interval I, if $F'(x) = f(x)$ for all x in I.

We can sometimes guess at antiderivatives, but as with limits and differentiation, our methods of calculation will become more sophisticated when we prove a few general rules for calculation.

Example Find an antiderivative for the function

$$f(x) = x^3,$$

Find another antiderivative for $f(x) = x^3$.

Although an antiderivative for a function is not unique, any two differ only by a constant. Recall that when we studied the Mean Value Theorem, we got the following two results which gives us some insight into how antiderivatives behave.

Theorem 1 If F is an antiderivative of the function $f(x) = 0$ on an interval I, then $F(x) = c$ for all x in I, where c is some constant.

Theorem 2 If F is an antiderivative for f, then every antiderivative for F is of the form $F(x) + c$, where c is a constant.

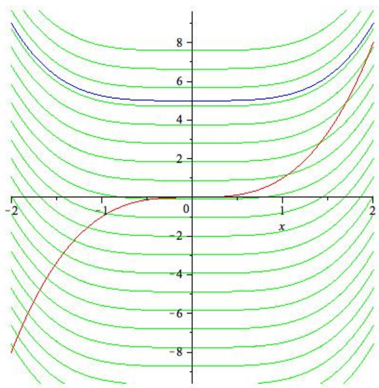
Note

- We often write **the general antiderivative of** f as $F(x)$ + C, where C denotes any constant. This really **refers to a family of functions**, one for each value of c , $F(x), F(x) + 1, F(x) + 50, F(x) - 1, F(x) - 2.3576542, etc....$ all of which are antiderivatives for f. Also, by the above theorem, every possible antiderivative for f is on this list.
- The traditional notation for the general antiderivative of a function $f(x)$ is $\displaystyle\int f(x)dx$.
- We will see below that if we specify the value of the antiderivative F at a particular value of x, say $F(0) = 1$, then only one of the antiderivatives from the list will have that property.

Example (a) Find the general antiderivative of $f(x) = x^3$.

(b) Find $F(x)$ for which $F'(x) = x^3$ and $F(0) = 5$.

The following graph shows a number of the antiderivatives of $f(x) = x^3$ with the graph of f shown in red and the solution to part b above shown in blue.



By reversing the rules of differentiation, we get the following rules for calculating antiderivatives:
If $F(x)$ is an antiderivative for $f(x)$ and $G(x)$ is an antiderivative for $g(x)$, then

1. $cF(x)$ is an antiderivative for $cf(x)$ where c is any constant.
2. $F(x) + G(x)$ is an antiderivative for $f(x) + g(x)$.

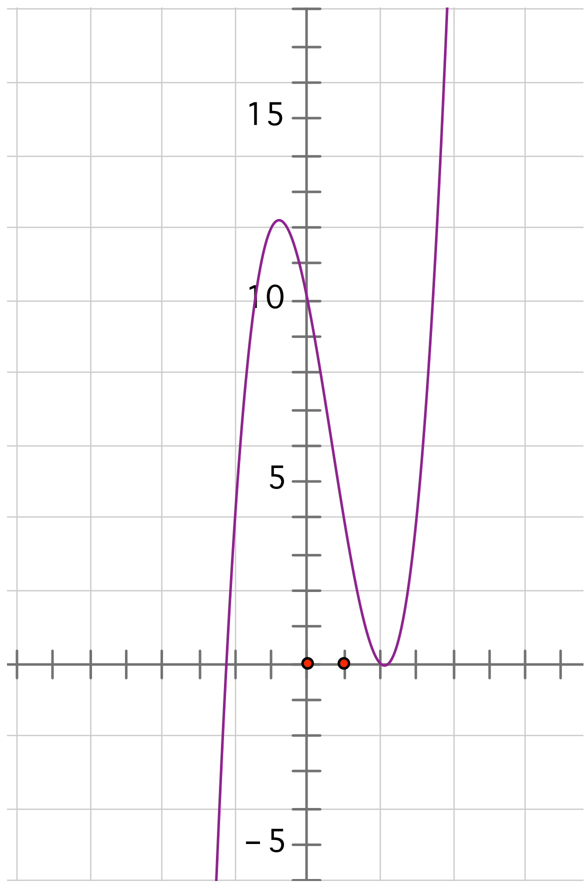
The following table shows some common antiderivatives which we can derive by reversing our work on differentiation:

Function $f(x)$	General Antiderivative $F(x) + C = \int f(x)dx$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\sec x \tan x$	$\sec x + C$

Example Find all functions F such that $F'(x) = 3\cos x + \sqrt[3]{x} + 4x^2$

Example Find $G(x)$ if $G'(x) = x^2 + 4x + 1$ and $G(0) = 10$.

Example The graph of a function $f(x)$ is shown in the diagram below, sketch the graph of an antiderivative of f.



Example A particle moving in a straight line has acceleration $a(t) = \cos t + \sin t$ where t is measured in seconds. We have the initial position is $s(0) = 0$ and initial velocity $v(0) = 5$ ft/s. Find the position function of the particle.

Example A ball is thrown vertically upwards with a speed of 10 ft/s from the edge of a cliff which is 400 ft. above the beach at its base. Let $h(t)$ denote its height in feet above the beach below t seconds later. When does the rock reach its maximum height and when does it hit the beach below? (Note acceleration due to gravity is $-32ft/s^2$).

Solution Let $h(t)$ denote the height of the ball above the beach below at t seconds after it is thrown. After the ball is thrown, the only force acting on it is the force of gravity, therefore we know that

$$\boxed{h''(t) = -32 \text{ ft}/s^2.}$$

From the information given, we also have that $\boxed{h(0) = 400}$ and $\boxed{h'(0) = 10}$.
Now we know that $h'(t)$ is an antiderivative for $a(t) = -32$ ft$/s^2$. Since

$$\int -32dt = -32t + C,$$

we must have that $h'(t) = -32t + C$ and since $h'(0) = C = 10$, we must have

$$h'(t) = -32t + 10.$$

The rock reached MAXIMUM HEIGHT when $h'(t) = 0$, that is when $10 = 32t$ or $t \approx 0.3125$ seconds. To find when the rock hits the beach below, we must find $h(t)$. This is an antiderivative for $h'(t)$.

$$\int h'(t)dt = \int (-32t + 10)dt = -16t^2 + 10t + D,$$

where D is a constant. Therefore $h(t) = -16t^2 + 10t + D$, for some constant D and since $h(0) = 400$, we have $D = 400$ and

$$h(t) = -16t^2 + 10t + 400.$$

The Rock HITS THE BEACH when $h(t) = 0$, that is when $-16t^2 + 10t + 400 = 0$ or

$$t = \frac{-10 \pm \sqrt{25700}}{-32} = 5.323 \text{ or } -4.64.$$

Since we are unable to throw rocks back in time, we have $\boxed{t = 5.323}$.

Extra Example A car driver fully applies the brakes producing a constant deceleration of $22 ft/s^2$ and producing skid marks (in a straight line) measuring 200 ft. as it comes to a halt. How fast was the car traveling when the brakes were first applied?

Please attempt this problem before you look at the solution on the next page.

Solution This car travels in a straight line, from the time the brakes are hit.
Let $S(t)$ denote the distance the car has travelled in feet t seconds after the brakes have been hit.
From the information given, we know that

$$\boxed{s(0) = 0} \text{ and } \boxed{s(t_1) = 200},$$

where t_1 is the number of seconds it takes for the car to come to a halt.
We let $v(t)$ denote the velocity of the car at time t and $a(t)$, the acceleration at time t. From the information given, we have

$$v(t_1) = 0 \quad \text{and} \quad a(t) = -22ft/s^2$$

for $0 \le t \le t_1$.

$$\boxed{\text{We want to find } v(0).}$$

Since $a(t) = v'(t)$, we have

$$v(t) = \int a(t)dt = \int (-22)dt = -22t + C.$$

Since $v(t_1) = 0$, we have $\boxed{C = 22t_1}$.
Since $s'(t) = v(t)$, we get

$$s(t) = \int v(t)dt = \int (-22t + C)dt = -11t^2 + Ct + D,$$

where D is a constant. Now since $s(0) = 0$, we get $\boxed{D = 0}$ and

$$s(t) = -11t^2 + Ct.$$

Using the fact that $C = 22t_1$ and $s(t_1) = 200$, we get

$$s(t_1) = -11t_1^2 + 22t_1^2 = 200 \quad \text{and} \quad 11t_1^2 = 200.$$

This gives that $\boxed{t_1 = \sqrt{\frac{200}{11}}}$ and

$$v(0) = C = 22t_1 = 22\sqrt{\frac{200}{11}} \approx 93.81 \; ft/s \approx 63.28 \; m.p.h.$$