Lecture 9 : Derivatives of Trigonometric Functions

(Please review Trigonometry under Algebra/Precalculus Review on the class webpage.)

In this section we will look at the derivatives of the trigonometric functions

$$\sin x, \quad \cos x, \quad \tan x \quad , \sec x, \quad \csc x, \quad \cot x.$$

Here the units used are radians and $\sin x = \sin(x \text{ radians})$. Recall that $\sin x$ and $\cos x$ are defined and continuous everywhere and

$$\tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{\cos x}{\sin x},$$

are continuous on their domains (all values of x where the denominator is non-zero). The graphs of the above functions are shown at the end of this lecture to help refresh your memory: Before we calculate the derivatives of these functions, we will calculate two very important limits.

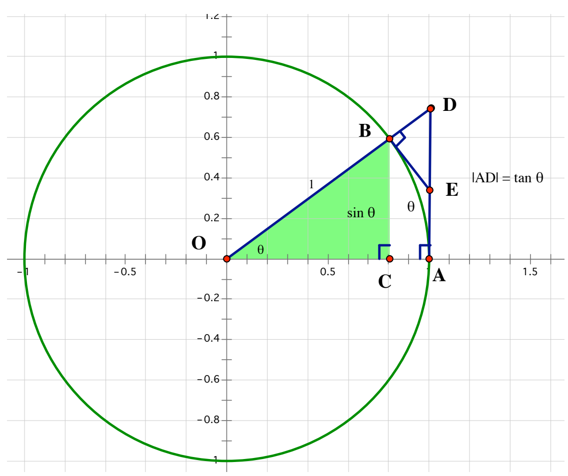
First Important Limit

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.}$$

See the end of this lecture for a geometric proof of the inequality,

$$\sin \theta < \theta < \tan \theta.$$

shown in the picture below for $\theta > 0$,



From this we can easily derive that

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

and we can use the squeeze theorem to prove that the limit shown above is 1.

Another Important Limit

From the above limit, we can derive that :

$$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$$

Example Calculate the limits:

$$\lim_{x \to 0} \frac{\sin 5x}{\sin 3x}, \qquad \lim_{x \to 0} \frac{\sin(x^3)}{x}.$$

Derivatives of Trigonometric Functions

1. From our trigonometric identities, we can show that $\boxed{\frac{d}{dx} \sin x = \cos x}$:

$$\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} =$$

$$\lim_{h \to 0} \frac{\sin(x)[\cos(h) - 1] + \cos(x)\sin(h)}{h} = \lim_{h \to 0} \sin(x) \frac{[\cos(h) - 1]}{h} + \lim_{h \to 0} \cos(x) \frac{\sin(h)}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{[\cos(h) - 1]}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x).$$

2. We can also show that $\boxed{\frac{d}{dx} \cos x = -\sin(x)}$:

$$\frac{d}{dx} \cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\cos(x)[\cos(h)-1]}{h} - \lim_{h\to 0} \frac{\sin(x)\sin(h)}{h}$$

$$= \cos(x)\lim_{h\to 0} \frac{[\cos(h)-1]}{h} - \sin(x)\lim_{h\to 0} \frac{\sin(h)}{h} = -\sin(x).$$

3. Using the derivatives of $\sin(x)$ and $\cos(x)$ and the quotient rule, we can deduce that $\boxed{\frac{d}{dx}\tan x = \sec^2(x)}$:

Example Find the derivative of the following function:

$$g(x) = \frac{1+\cos x}{x+\sin x}$$

Higher Derivatives

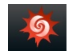
We see that the higher derivatives of $\sin x$ and $\cos x$ form a pattern in that they repeat with a cycle of four. For example, if $f(x) = \sin x$, then

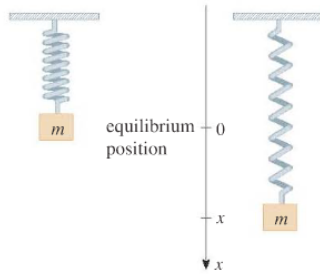
$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f^{(3)}(x) = -\cos x, \quad f^{(4)}(x) = \sin x, \quad f^{(5)}(x) = \cos x, \ldots$$

(Note the derivatives follow a similar pattern for $\cos(x)$.)

Example Let $f(x) = \sin x$. What is

$$f^{(20)}(x)?$$

A mass on a spring released at some point other than its equilibrium position will follow a pattern of simple harmonic motion ($x(t) = A\sin(Cx + D)$ or equivalently $x(t) = A\cos(Cx + D)$), when there is no friction or other forces to dampen the effect. The values of A, C and D depend on the elasticity of the spring, the mass and the point at which the mass is released. You will be able to prove this easily later when you learn about differential equations. 



Example An object at the end of a vertical spring is stretched 5cm beyond its rest position and released at time $t = 0$. Its position at time t is given by $x(t)$ with the positive direction as shown in a downward direction, where

$$x(t) = 5\cos(t).$$

(a) Find the velocity and acceleration at time t.

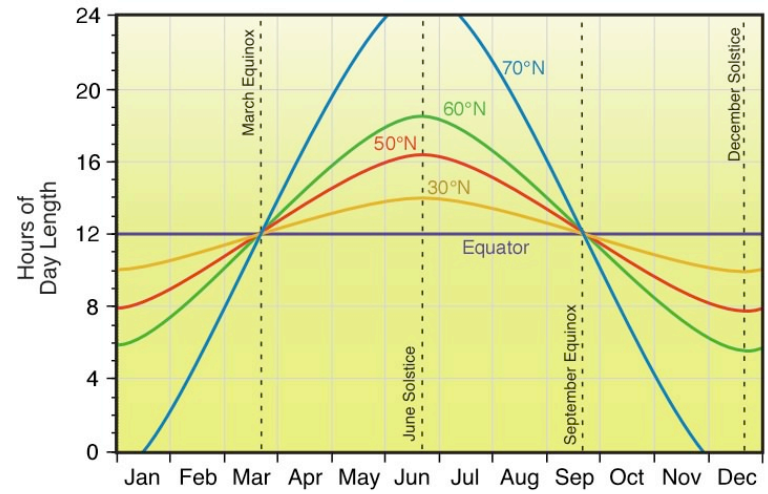
(b) Find the position, velocity and acceleration of the mass at time $t = \frac{\pi}{4}$. In which direction is it moving at that time?

The following is a summary of the derivatives of the trigonometric functions. You should be able to verify all of the formulas easily.

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x, \qquad \frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Example The graph below shows the variations in day length for various degrees of Lattitude.

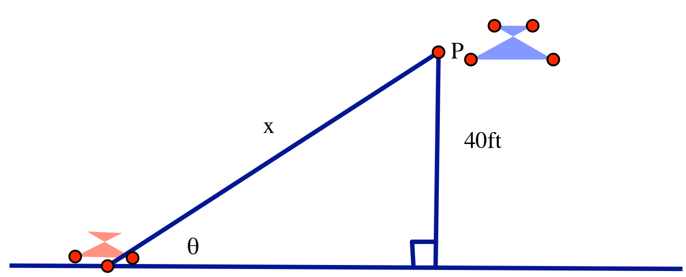


i-2: Annual variations in day length for locations at the equator, 30, 50, 60, and 70° North latitude.

At 60^o North, at what times of the year is the length of the day changing most rapidly?

Extras

Example (Preparation for Related Rates) A police car is parked 40 feet from the road at the point P in the diagram below. Your vehicle is approaching on the road as in the diagram below and the police are pointing a radar gun at your car. Let x denote the distance from your car to the police car and let θ be the angle between the line of sight of the radar gun and the road. How fast is x changing with respect to θ when $\theta = \frac{\pi}{4}$? (Please attempt this problem before looking at the solution on the following page.)



Solution We have that the variables x and θ are related in the following way:

$$\frac{40}{x} = \sin(\theta).$$

Therefore

$$\frac{40}{\sin(\theta)} = x$$

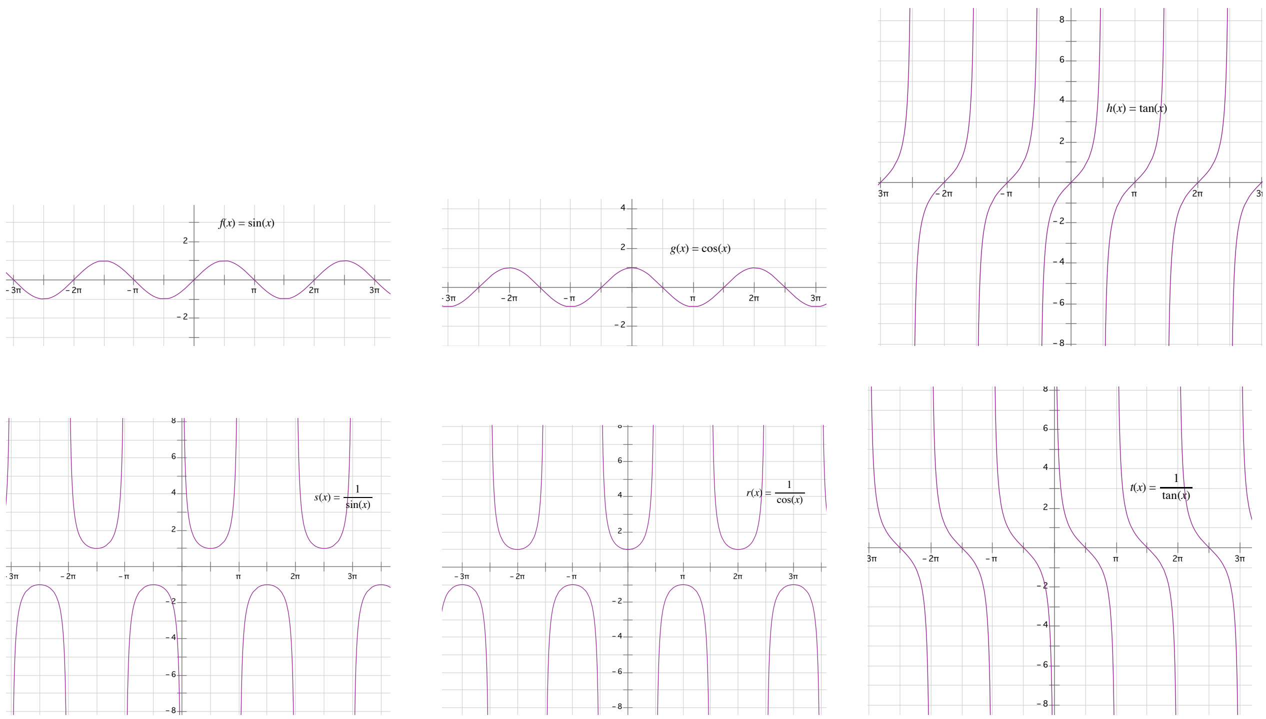
and

$$\frac{dx}{d\theta} = 40\Big[\frac{-\cos(\theta)}{\sin^2(\theta)}\Big].$$

When $\theta = \frac{\pi}{4}$,

$$\frac{dx}{d\theta}\Big|_{\theta=\frac{\pi}{4}} = 40\Big[\frac{-\cos(\frac{\pi}{4})}{\sin^2(\frac{\pi}{4})}\Big] = 40\frac{-1/\sqrt{2}}{1/2} = -40\sqrt{2} \quad \text{feet per radian.}$$

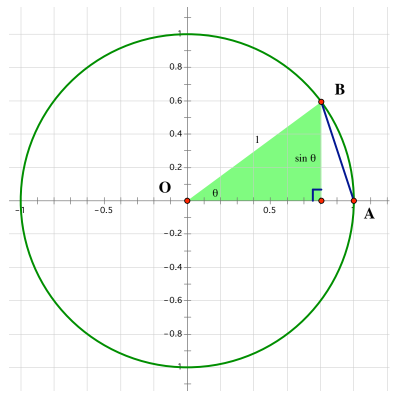
Graphs of Trigonometric functions



Inequality

Let θ be an angle close to 0, and between 0 and $\frac{\pi}{2}$. Note that since $\sin\theta = -\sin(-\theta)$, we have $\frac{\sin\theta}{\theta} = \frac{\sin(-\theta)}{-\theta}$ and $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^-} \frac{\sin\theta}{\theta}$. Because of this, we need only consider the right hand limit, $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$ with $\theta > 0$.

In the picture below, we see that θ, which is the length of the arc of the unit circle from A to B in larger than the length of the line segment from A to B. The line segment from A to B is larger than $\sin\theta$ since it is the hypotenuse of a right triangle with a side of length $\sin\theta$.



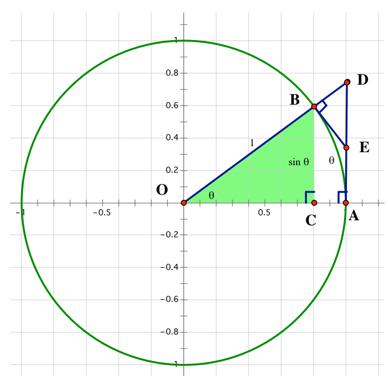
From this we can conclude that $\sin\theta < \theta$ or

$$\frac{\sin\theta}{\theta} < 1.$$

Now consider the picture below. We can see intuitively that the length of the arc of the unit circle from A to B is smaller than the sum of the lengths of the line segments $|AE| + |EB|$. Because the line segment EB is a side of a right triangle with hypotenuse ED, we see that $|EB| < |ED|$. Thus we have

$$\theta < |AE| + |EB| < |AE| + |ED| = |AD|$$

Note now that $\frac{|AD|}{|OA|} = \tan\theta$ and $|AD| = |OA|\tan\theta = \tan\theta$.



We now have that

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{giving} \quad \cos\theta < \frac{\sin\theta}{\theta}$$

since $\cos\theta > 0$ (when we multiply by positive numbers, inequalities are preserved).

Putting both inequalities together we get

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Extra Problems

1. Calculate
$$\lim_{x \to o} \frac{\sin(x^3)}{x}.$$

2. Calculate
$$\lim_{x \to 0} 7x \cot(3x).$$

3. If $g(x) = \cos(x)$, what is $g^{(42)}(x)$?

4. Find $f'(x)$ if $f(x) = x^2 \cos(x) \sin(x)$.

Extra Problems : Solutions

1. Calculate

$$\lim_{x\to o} \frac{\sin(x^3)}{x}.$$

$\lim_{x\to o} \frac{\sin(x^3)}{x} = \lim_{x\to 0} \frac{\sin(x^3)}{x\cdot x^2} \cdot x^2 = \lim_{x\to 0} \frac{\sin(x^3)}{x^3} \cdot \lim_{x\to 0} x^2 = 1 \cdot 0 = 0.$

2. Calculate

$$\lim_{x\to 0} 7x \cot(3x).$$

$$\lim_{x\to 0} 7x\cot(3x) = \lim_{x\to 0} 7x\frac{\cos(3x)}{\sin(3x)} = \lim_{x\to 0} 7x \cdot \frac{3x}{3x} \cdot \frac{\cos(3x)}{\sin(3x)} =$$

$$\lim_{x\to 0} 7 \lim_{x\to 0} \frac{(3x)}{\sin(3x)} \lim_{x\to 0} \frac{\cos(3x)}{3} = 7 \cdot 1 \cdot 0 = 0$$

3. If $g(x) = \cos(x)$, what is $g^{(42)}(x)$?

$$g'(x) = -\sin x, \quad g''(x) = -\cos x, \quad g^{(3)}(x) = \sin x, \quad g^{(4)}(x) = \cos x, \ldots$$

Therefore $g^{(40)}(x) = \cos x$ and $g^{(42)}(x) = -\cos x$.

4. Find $f'(x)$ if $f(x) = x^2 \cos(x)\sin(x)$.

Using the product rule, we get

$$f'(x) = (\cos x \cdot \sin x)2x + x^2 \frac{d}{dx}(\cos x \cdot \sin x)$$

using the quotient rule a second time, we get

$$f'(x) = 2x(\cos x \cdot \sin x) + x^2(\sin x(-\sin x) + \cos x \cos x) = 2x(\cos x \cdot \sin x) + x^2(\cos^2 x - \sin^2 x).$$

In fact if we know our trig formulas very well, we see that

$$f'(x) = x\sin(2x) + x^2(\cos(2x)).$$