Lecture 4 : Calculating Limits using Limit Laws

Click on this symbol to view an interactive demonstration in Wolfram Alpha.

Using the definition of the limit, $\lim_{x\to a} f(x)$, we can derive many general laws of limits, that help us to calculate limits quickly and easily. The following rules apply to any functions $f(x)$ and $g(x)$ and also apply to left and right sided limits:

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist (meaning they are finite numbers). Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$;
   (the limit of a sum is the sum of the limits).
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$;
   (the limit of a difference is the difference of the limits).
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$;
   (the limit of a constant times a function is the constant times the limit of the function).
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$;
   (The limit of a product is the product of the limits).
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$;
   (the limit of a quotient is the quotient of the limits provided that the limit of the denominator is not 0)

Example If I am given that

$$\lim_{x\to 2} f(x) = 2, \qquad \lim_{x\to 2} g(x) = 5, \qquad \lim_{x\to 2} h(x) = 0.$$

find the limits that exist (are a finite number):

$$(a)\quad \lim_{x\to 2}\frac{2f(x)+h(x)}{g(x)} = \frac{\lim_{x\to 2}(2f(x)+h(x))}{\lim_{x\to 2} g(x)} \quad \text{since } \lim_{x\to 2} g(x) \neq 0$$

$$= \frac{2\lim_{x\to 2} f(x) + \lim_{x\to 2} h(x)}{\lim_{x\to 2} g(x)} = \frac{2(2)+0}{5} = \frac{4}{5}$$

$$(b)\quad \lim_{x\to 2}\frac{f(x)}{h(x)} \qquad\qquad (c)\quad \lim_{x\to 2}\frac{f(x)h(x)}{g(x)}$$

Note 1 If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) = b$, where b is a finite number with $b \neq 0$, Then: the values of the quotient $\frac{f(x)}{g(x)}$ can be made arbitrarily large in absolute value as $x \to a$ and thus

the limit does not exist.
If the values of $\frac{f(x)}{g(x)}$ are positive as $x \to a$ in the above situation, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$,
If the values of $\frac{f(x)}{g(x)}$ are negative as $x \to a$ in the above situation, then $\lim_{x\to a} \frac{f(x)}{g(x)} = -\infty$,
If on the other hand, if $\lim_{x\to a} g(x) = 0 = \lim_{x\to a} f(x)$, we cannot make any conclusions about the limit.

Example Find $\lim_{x\to\pi^-} \frac{\cos x}{x-\pi}$.

As x approaches π from the left, $\cos x$ approaches a finite number -1.

As x approaches π from the left, $x - \pi$ approaches 0.

Therefore as x approaches π from the left, the quotient $\frac{\cos x}{x-\pi}$ approaches ∞ in absolute value.

The values of both $\cos x$ and $x - \pi$ are negative as x approaches π from the left, therefore

$$\lim_{x\to\pi^-} \frac{\cos x}{x-\pi} = \infty.$$

More powerful laws of limits can be derived using the above laws 1-5 and our knowledge of some basic functions. The following can be proven reasonably easily (we are still assuming that c is a constant and $\lim_{x\to a} f(x)$ exists);

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer (we see this using rule 4 repeatedly).
7. $\lim_{x\to a} c = c$, where c is a constant (easy to prove from definition of limit and easy to see from the graph, $y = c$).
8. $\lim_{x\to a} x = a$, (follows easily from the definition of limit)
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer (this follows from rules 6 and 8).
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is a positive integer and $a > 0$ if n is even. (proof needs a little extra work and the binomial theorem)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ assuming that the $\lim_{x\to a} f(x) > 0$ if n is even. (We will look at this in more detail when we get to continuity)

Example Evaluate the following limits and justify each step:

(a) $\lim_{x\to 3} \frac{x^3+2x^2-x+1}{x-1}$

(b) $\lim_{x\to 1} \sqrt[3]{x+1}$

(c) Determine the infinite limit (see note 1 above, say if the limit is ∞, $-\infty$ or D.N.E.) $\lim_{x\to 2^-} \frac{x+1}{(x-2)}$.

Polynomial and Rational Functions

Please review the relevant parts of Lectures 3, 4 and 7 from the Algebra/Precalculus review page. This demonstration will help you visualize some rational functions:

Direct Substitution (Evaluation) Property If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$. This follows easily from the rules shown above. (Note that this is the case in part (a) of the example above)

if $f(x) = \frac{P(x)}{Q(x)}$ is a rational function where $P(x)$ and $Q(x)$ are polynomials with $Q(a) = 0$, then:
If $P(a) \neq 0$, we see from note 1 above that $\lim_{x\to a} \frac{P(x)}{Q(x)} = \pm\infty$ or D.N.E. and is not equal to $\pm\infty$.
If $P(a) = 0$ we can cancel a factor of the polynomial $P(x)$ with a factor of the polynomial $Q(x)$ and the resulting rational function may have a finite limit or an infinite limit or no limit at $x = a$. The limit of the new quotient as $x \to a$ is equal to $\lim_{x\to a} \frac{P(x)}{Q(x)}$ by the following observation which we made in the last lecture:

Note 2: If $h(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$ provided the limits exist.

Example Determine if the following limits are finite, equal to $\pm\infty$ or D.N.E. and are not equal to $\pm\infty$:

(a) $\lim_{x\to 3} \frac{x^2-9}{x-3}$.

(b) $\lim_{x\to 1^-} \frac{x^2-x-6}{x-1}$.

(c) Which of the following is true:
1. $\lim_{x\to 1} \frac{x^2-x-6}{x-1} = +\infty$, 2. $\lim_{x\to 1} \frac{x^2-x-6}{x-1} = -\infty$, 3. $\lim_{x\to 1} \frac{x^2-x-6}{x-1}$ D.N.E. and is not $\pm\infty$,

Example Evaluate the limit (finish the calculation)

$$\lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h}.$$

$\lim_{h \to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h \to 0} \frac{9+6h+h^2-9}{h} =$

Example Evaluate the following limit:

$$\lim_{x \to 0} \frac{\sqrt{x^2+25} - 5}{x^2}.$$

Recall also our observation from the last day which can be proven rigorously from the definition (this is good to keep in mind when dealing with piecewise defined functions):

Theorm $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Example Evaluate the limit if it exists:

$$\lim_{x \to -2} \frac{3x+6}{|x+2|}$$

The following theorems help us calculate some important limits by comparing the behavior of a function with that of other functions for which we can calculate limits:

Theorem If $f(x) \leq g(x)$ when x is near a(except possible at a) and the limits of $f(x)$ and $g(x)$ both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

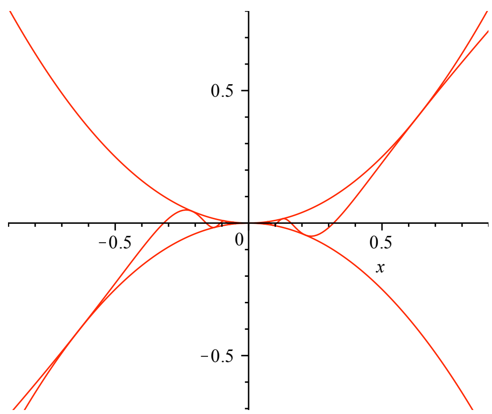
The Sandwich (squeeze) Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L.$$

Recall last day, we saw that $\lim_{x \to 0} \sin(1/x)$ does not exist because of how the function oscillates near $x = 0$. However we can see from the graph below and the above theorem that $\lim_{x \to 0} x^2 \sin(1/x) = 0$, since the graph of the function is sandwiched between $y = -x^2$ and $y = x^2$:



Example Calculate the limit $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.
We have $-1 \leq \sin(1/x) \leq 1$ for all x,
multiplying across by x^2 (which is positive), we get $-x^2 \leq x^2 \sin(1/x) \leq x^2$ for all x,
Using the Sandwich theorem, we get

$$0 = \lim_{x \to 0} -x^2 \leq \lim_{x \to o} x^2 \sin(1/x) \leq \lim_{x \to 0} x^2 = 0$$

Hence we can conclude that

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$

Example Decide if the following limit exists and if so find its values:

$\lim_{x \to o} x^{100} \cos^2(\pi/x)$

Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

(1) $\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3$

(2) $\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2}$.

(3) $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$.

(4) $\lim_{x\to 0} \frac{|x|}{x^2+x+10}$.

(5) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

(6) If $2x \leq g(x) \leq x^2 - x + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

(7) Determine if the following limit is finite, $\pm\infty$ or D.N.E. and is not $\pm\infty$.

$$\lim_{x\to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)}.$$

Extra Examples, attempt the problems before looking at the solutions

Decide if the following limits exist and if a limit exists, find its value.

(1) $\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3$

Since this is a polynomial function, we can calculate the limit by direct substitution:

$$\lim_{x\to 1} x^4 + 2x^3 + x^2 + 3 = 1^4 + 2(1)^3 + 1^2 + 3 = 7.$$

(2) $\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2}$.

This is a rational function, where both numerator and denominator approach 0 as x approaches 2. We factor the numerator to get

$$\lim_{x\to 2} \frac{x^2-3x+2}{(x-2)^2} = \lim_{x\to 2} \frac{(x-1)(x-2)}{(x-2)^2}$$

After cancellation, we get

$$\lim_{x\to 2} \frac{(x-1)(x-2)}{(x-2)^2} = \lim_{x\to 2} \frac{(x-1)}{(x-2)}.$$

Now this is a rational function where the numerator approaches 1 as $x \to 2$ and the denominator approaches 0 as $x \to 2$. Therefore

$$\lim_{x\to 2} \frac{(x-1)}{(x-2)}$$

does not exist.

We can analyze this limit a little further, by checking out the left and right hand limits at 2. As x approaches 2 from the left, the values of $(x-1)$ are positive (approaching a constant 1) and the values of $(x-2)$ are negative (approaching 0). Therefore the values of $\frac{(x-1)}{(x-2)}$ are negative and become very large in absolute value. Therefore

$$\lim_{x\to 2^-} \frac{(x-1)}{(x-2)} = -\infty.$$

Similarly, you can show that

$$\lim_{x\to 2^-} \frac{(x-1)}{(x-2)} = +\infty,$$

and therefore the graph of $y = \frac{(x-1)}{(x-2)}$ has a vertical asymptote at $x = 2$.
(check it out on your calculator)

(3) $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$.

Let $f(x) = \frac{1}{x} - \frac{1}{|x|}$. We write this function as a piecewise defined function:

$$f(x) = \begin{cases} \frac{1}{x} - \frac{1}{x} = 0 & x > 0 \\ \frac{1}{x} + \frac{1}{x} = \frac{2}{x} & x \le 0 \end{cases}.$$

$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$ exists only if the left and right hand limits exist and are equal.
$\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{|x|}) = \lim_{x\to 0^+} 0 = 0$ and $\lim_{x\to 0^-}(\frac{1}{x} - \frac{1}{|x|}) = \lim_{x\to 0^-} \frac{2}{x} = -\infty$.
Since the limits do not match, we have

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|}) \quad D.N.E.$$

(4) $\lim_{x\to 0} \frac{|x|}{x^2+x+10}$.

Since $\lim_{x\to 0} x^2 + x + 10 = 10 \neq 0$, we have

$$\lim_{x\to 0} \frac{|x|}{x^2+x+10} = \frac{\lim_{x\to 0} |x|}{\lim_{x\to 0}(x^2+x+10)} = \frac{\lim_{x\to 0} |x|}{10}.$$

Now

$$|x| = \begin{cases} x & x > 0 \\ -x & x \leq 0 \end{cases}.$$

Clearly $\lim_{x\to 0^+} |x| = 0 = \lim_{x\to 0^-} |x|$. Hence $\lim_{x\to 0} |x| = 0$ and

$$\lim_{x\to 0} \frac{|x|}{x^2+x+10} = \frac{\lim_{x\to 0} |x|}{10} = \frac{0}{10} = 0.$$

(5) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

Since $\lim_{h\to 0} \sqrt{4+h} - 2 = 0 = \lim_{h\to 0} h$, we cannot determine whether this limit exists or not from the limit laws without some transformation. We have

$$\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{(\sqrt{4+h})^2-4)}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{(\sqrt{4+h}+2)} = \frac{1}{4}.$$

(6) If $2x \leq g(x) \leq x^2 - x + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

We use the Sandwich theorem here. Since $2x \leq g(x) \leq x^2 - x + 2$, we have

$$\lim_{x\to 1} 2x \leq \lim_{x\to 1} g(x) \leq \lim_{x\to 1}(x^2 - x + 2),$$

therefore

$$2 \leq \lim_{x\to 1} g(x) \leq 2$$

and hence

$$\lim_{x\to 1} g(x) = 2.$$

(7) Determine if the following limit is finite, $\pm\infty$ or D.N.E. and is not $\pm\infty$.

$$\lim_{x\to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)}.$$

Let $P(x) = (x-3)(x+2)$ and $Q(x) = (x-1)(x-2)$. We have $P(1) = -6 \neq 0$ and $Q(1) = 0$. Therefore the values of $\frac{P(x)}{Q(x)} = \frac{(x-3)(x+2)}{(x-1)(x-2)}$ get larger in absolute value as x approaches 1.

As x approaches 1 from the left, $(x-3) < 0$, $(x-2) < 0$, $(x-1) < 0$, and $(x+2) > 0$, therefore the quotient $\frac{(x-3)(x+2)}{(x-1)(x-2)} < 0$ as x approaches 1 from the left and therefore

$$\lim_{x\to 1^-} \frac{(x-3)(x+2)}{(x-1)(x-2)} = -\infty.$$