Lecture 1 : Inverse functions

One-to-one Functions A function \( f \) is one-to-one if it never takes the same value twice or

\[ f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2. \]

**Example** The function \( f(x) = x \) is one to one, because if \( x_1 \neq x_2 \), then \( f(x_1) \neq f(x_2) \).

On the other hand the function \( g(x) = x^2 \) is not a one-to-one function, because \( g(-1) = g(1) \).

**Graph of a one-to-one function** If \( f \) is a one to one function then no two points \((x_1, y_1), \ (x_2, y_2)\) have the same \( y \)-value. Therefore no horizontal line cuts the graph of the equation \( y = f(x) \) more than once.

**Example** Compare the graphs of the above functions

![Graphs of functions](image)

**Determining if a function is one-to-one**

**Horizontal Line test:** A graph passes the Horizontal line test if each horizontal line cuts the graph at most once.

**Using the graph to determine if \( f \) is one-to-one**
A function \( f \) is one-to-one if and only if the graph \( y = f(x) \) passes the Horizontal Line Test.

**Example** Which of the following functions are one-to-one?

![Graphs of functions](image)

**Using the derivative to determine if \( f \) is one-to-one**
A function whose derivative is always positive or always negative is a one-to-one function. Why?

**Example** Is the function \( g(x) = \sqrt{4x + 4} \) a one-to-one function?
Inverse functions

**Inverse Functions** If $f$ is a one-to-one function with domain $A$ and range $B$, we can define an inverse function $f^{-1}$ (with domain $B$) by the rule

\[ f^{-1}(y) = x \text{ if and only if } f(x) = y. \]

This is a sound definition of a function, precisely because each value of $y$ in the domain of $f^{-1}$ has exactly one $x$ in $A$ associated to it by the rule $y = f(x)$.

**Example** If $f(x) = x^3 + 1$, use the equivalence of equations given above find $f^{-1}(9)$ and $f^{-1}(28)$.

**Note** that the domain of $f^{-1}$ equals the range of $f$ and the range of $f^{-1}$ equals the domain of $f$.

**Example** Let $g(x) = \sqrt{4x+4}$. What is Domain $f$?
What is Range $g$?
Does $g^{-1}$ exist?
What is Domain $g^{-1}$?
What is Range $g^{-1}$?
What is $g^{-1}(4)$?

**Finding a Formula For $f^{-1}(x)$**

Given a formula for $f(x)$, we would like to find a formula for $f^{-1}(x)$. Using the equivalence

\[ x = f^{-1}(y) \text{ if and only if } y = f(x) \]

we can sometimes find a formula for $f^{-1}$ using the following method:

1. In the equation $y = f(x)$, if possible solve for $x$ in terms of $y$ to get a formula $x = f^{-1}(y)$.
2. Switch the roles of $x$ and $y$ to get a formula for $f^{-1}$ of the form $y = f^{-1}(x)$.

**Example** Let $f(x) = \frac{2x+1}{x-3}$, find a formula for $f^{-1}(x)$. 
Composing $f$ and $f^{-1}$.

We have

$$\text{if } x = f^{-1}(y) \text{ then } y = f(x).$$

Substituting $f(x)$ for $y$ in the equation on the left, we get

$$f^{-1}(f(x)) = x.$$ 

Similarly

$$\text{if } x = f(y) \text{ then } y = f^{-1}(x)$$

and substituting $f^{-1}(x)$ for $y$ in the equation on the left, we get

$$f(f^{-1}(x)) = x.$$ 

**Example** Above, we found that if $f(x) = \frac{2x+1}{x-3}$, then $f^{-1}(x) = \frac{3x+1}{x-2}$. We can check the above formula for the composition:

$$f(f^{-1}(x)) = f\left(\frac{3x + 1}{x-2}\right) = 2\left(\frac{\frac{3x+1}{x-2}+1}{\frac{3x+1}{x-2}}\right) - 3 = \frac{(6x + 2 + x - 2)/(x-2)}{(3x + 1 - 3x + 6)/(x-2)} = \frac{7x}{7} = x.$$ 

You should also check that $f^{-1}(f(x)) = x$.

**Graph of $y = f^{-1}(x)$**

Since the equation $y = f^{-1}(x)$ is the same as the equation $x = f(y)$, the graphs of both equations are identical. To graph the equation $x = f(y)$, we note that this equation results from switching the roles of $x$ and $y$ in the equation $y = f(x)$. This transformation of the equation results in a transformation of the graph amounting to reflection in the line $y = x$. Thus

the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$ and vice versa.

**Note** The reflection of the point $(x_1, y_1)$ on the line $y = x$ is $(y_1, x_1)$. Therefore if the point $(x_1, y_1)$ is on the graph of $y = f^{-1}(x)$, we must have $(y_1, x_1)$ on the graph of $y = f(x)$.

The graphs of $f(x) = \frac{2x+1}{x-3}$ and $f^{-1}(x) = \frac{3x+1}{x-2}$ are shown below.
We can derive properties of the graph of \( y = f^{-1}(x) \) from properties of the graph of \( y = f(x) \), since they are reflections of each other in the line \( y = x \). For example:

**Theorem** If \( f \) is a one-to-one continuous function defined on an interval, then its inverse \( f^{-1} \) is also one-to-one and continuous. (Thus \( f^{-1}(x) \) has an inverse, which has to be \( f(x) \), by the equivalence of equations given in the definition of the inverse function.)

**Theorem** If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.
\]

**proof** \( y = f^{-1}(x) \) if and only if \( x = f(y) \). Using implicit differentiation we differentiate \( x = f(y) \) with respect to \( x \) to get

\[
1 = f'(y) \frac{dy}{dx} \quad \text{or} \quad \frac{1}{f'(y)} = \frac{dy}{dx}.
\]

or

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = (f^{-1})'(x)
\]

**Geometrically** this means that if \((a, f^{-1}(a))\) is a point on the curve \( y = f^{-1}(x) \), then the point \((f^{-1}(a), a)\) is on the curve \( y = f(x) \) and the slope of the tangent to the curve \( y = f^{-1}(x) \) at \((a, f^{-1}(a))\) is the reciprocal of the tangent to the curve \( y = f(x) \) at the point \((f^{-1}(a), a)\). The graphs of the function \( f(x) = \frac{2x+1}{x-3} \) and \( f^{-1}(x) = \frac{3x+1}{x-2} \) are shown below. You can verify that \(-7 = (f^{-1})'(3) = \frac{1}{f'(10)}\).

![Graph of functions and their inverses](image)

**Note** To use the above formula for \((f^{-1})'(a)\), you do not need the formula for \(f^{-1}(x)\), you only need the value of \(f^{-1}\) at \(a\) and the value of \(f\) at \(f^{-1}(a)\).

**Example** Consider the function \( f(x) = \sqrt{4x+4} \) defined above. Find \((f^{-1})'(4)\).

What does the formula from the theorem say?

Use the equivalence of the equations \( y = f^{-1}(x) \) and \( x = f(y) \) to find \(f^{-1}(4)\).

Put this in the formula from the theorem to find \((f^{-1})'(4)\).
Example Let \( f(x) = x^3 + 1 \), find \( (f^{-1})'(28) \).

Example If \( f \) is a one-to-one function with the following properties:

\[
f(10) = 21, \quad f'(10) = 2, \quad f^{-1}(10) = 4.5, \quad f'(4.5) = 3.
\]

Find \( (f^{-1})'(10) \).