Lecture 10 : Trigonometric Substitution

To solve integrals containing the following expressions:

\[ \sqrt{a^2 - x^2}, \quad \sqrt{x^2 + a^2}, \quad \sqrt{x^2 - a^2}, \]

it is sometimes useful to make the following substitutions:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} ) or ( \theta = \sin^{-1} \frac{x}{a} )</td>
<td>( 1 - \sin^2 \theta = \cos^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} ) or ( \theta = \tan^{-1} \frac{x}{a} )</td>
<td>( 1 + \tan^2 \theta = \sec^2 \theta )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta, \quad 0 \leq \theta &lt; \frac{\pi}{2} ) or ( \pi \leq \theta &lt; \frac{3\pi}{2} ) or ( \theta = \sec^{-1} \frac{x}{a} )</td>
<td>( \sec^2 \theta - 1 = \tan^2 \theta )</td>
</tr>
</tbody>
</table>

**Note** The calculations here are much easier if you use the substitution in reverse: \( x = a \sin \theta \) as opposed to \( \theta = \sin^{-1} \frac{x}{a} \).

**Integrals involving \( \sqrt{a^2 - x^2} \)**

We make the substitution \( x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), \( dx = a \cos \theta d\theta \),

\[ \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a|\cos \theta| = a \cos \theta \quad (\text{since } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ by choice}) \]

**Example**

\[ \int \frac{x^3}{\sqrt{4 - x^2}} \, dx \]

**Example**

\[ \int \frac{dx}{x^2 \sqrt{9 - x^2}} \]

You can use this method to derive what you already know

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C \]
Integrals involving $\sqrt{x^2 + a^2}$

We make the substitution $x = a \tan \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $dx = a \sec^2 \theta d\theta$,

$$\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2 \theta + a^2} = a |\sec \theta| = a \sec \theta \text{ (since } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ by choice. })$$

Example

$$\int \frac{dx}{\sqrt{x^2 + 4}}$$

You can also use this substitution to get the familiar

$$\int \frac{1}{x^2 + a^2}dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$  

Completing The Square

Sometimes we can convert an integral to a form where trigonometric substitution can be applied by completing the square.

Example  Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 4x + 8}}.$$
Integrals involving $\sqrt{x^2 - a^2}$

We make the substitution $x = a \sec \theta$, $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, $dx = a \sec \theta \tan \theta d\theta$.

This amounts to saying $\theta = \sec^{-1} \frac{x}{a}$.

$$
\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = a |\tan \theta| = a \tan \theta \text{ (since } 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2} \text{ by choice.)}
$$

**Example**  Evaluate

$$
\int \frac{1}{x^2 \sqrt{x^2 - a^2}} dx
$$

**Example**  Evaluate

$$
\int_{4}^{6} \frac{1}{\sqrt{x^2 - 6x + 8}} dx
$$

You can also use this substitution to get

$$
\int \frac{1}{x \sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + C.
$$