Lecture 11/12 : Partial Fractions

In this section we look at integrals of rational functions.

Essential Background

A **Polynomial** $P(x)$ is a linear sum of powers of x, for example $3x^3 + 3x^2 + x + 1$ or $x^5 - x$.

The **degree** of a polynomial $P(x)$ is the highest power occurring in the polynomial, for example the degree of $3x^3 + 3x^2 + x + 1$ is 3 and the degree of $x^5 - x$ is 5.

Fundamental Theorem of Algebra Every polynomial can be factored into linear factors of the form $ax + b$ and irreducible quadratic factors ($ax^2 + bx + c$ where $b^2 - 4ac < 0$) where a, b and c are constants.

For example $3x^3 + 3x^2 + x + 1 = (x + 1)(3x^2 + 1)$ and
$x^5 - x = x(x^4 - 1) = x(x^2 - 1)(x^2 + 1) = x(x - 1)(x + 1)(x^2 + 1)$.

A **Rational Function** is a quotient of 2 polynomials $\frac{P(x)}{Q(x)}$.

The rational function $\frac{R(x)}{Q(x)}$ is **a proper rational function** is $DegR(x) < DegQ(x)$. In this case, we can write the rational function as a sum of **Partial Fractions** of the form

$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^i}$$

where A and B are constants and i is a non-negative integer.

We already know how to integrate these partial fractions.using substitution, trigonometric substitution or logarithms. We will go through the method of solving for the constants in the partial fraction expansion of a proper rational function in steps.

Step 1 : The Denominator $Q(x)$ is a product of distinct linear factors

If $Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_1x + b_1)$ we include a quotient of the form $\frac{A_i}{(a_ix+b_i)}$ for each term in the partial fraction expansion. We write

$$\frac{R(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \cdots + \frac{A_n}{(a_nx+b_n)}$$

and solve for $A_1, A_2, \dots, A_n$ by multiplying this equation by the lowest common denominator of the Right Hand Side which is the product of the linear factors $Q(x)$.

Note: we know how to evaluate the integral $\int \frac{A_i}{(a_ix+b_i)}dx$ using a substitution and logarithms.

Example Evaluate

$$\int \frac{1}{x^2-25}dx$$

Step 2 The denominator has repeated linear factors, that is factors of the form $(a_i x + b_i)^k$ where $k > 1$.

For every factor of type $(a_i x + b_i)^k$ in the denominator we include a sum of type

$$\frac{A_1}{(a_i x + b_i)} + \frac{A_2}{(a_i x + b_i)^2} + \cdots + \frac{A_n}{(a_i x + b_i)^k}$$

in the partial fractions decomposition of the rational function.

For Example, the partial fractions expansion of

$$\frac{x^3 + 2x + 2}{(x-2)^3(x-1)^2}$$

looks like

$$\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2}$$

Note that we can integrate all of these partial fractions using logarithms or integration of powers.

Example Evaluate

$$\int \frac{2x+4}{x^3 - 2x^2} dx$$

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Step 3: The denominator $Q(x)$ has factors which are irreducible quadratics, none of which are repeated,
that is factors of the form $a_i x^2 + b_i x + c_i$ where $b_i^2 - 4a_i c_i < 0$.

In this case we include a term of the form

$$\frac{A_i x + B_i}{a_i x^2 + b_i x + c_i}$$

in the partial fractions decomposition for each such factor. Note that we can integrate this using a combination of substitution and the fact that

$$\boxed{\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}.$$

Example Evaluate

$$\int \frac{(x^2 + x + 1)}{x(x^2 + 1)} dx$$

Step 4: When the denominator $Q(x)$ has repeated irreducible quadratic factors of the form $(a_ix^2 + b_ix + c_i)^k$ we include a sum of type

$$\frac{A_1x + B_1}{(a_ix^2 + b_ix + c_i)} + \frac{A_2x + B_2}{(a_ix^2 + b_ix + c_i)^2} + \cdots + \frac{A_nx + B_n}{(a_ix^2 + b_ix + c_i)^k}$$

in the partial fractions expansion of the rational function for each such term.

Note to integrate expressions like $\frac{A_nx+B_n}{(a_ix^2+b_ix+c_i)^k}$, we may be able to use regular substitution, or we may have to use a trigonometric substitution.

Example Evaluate

$$\int \frac{x^4 + x^2 + 1}{x(x^2 + 1)^2} dx.$$

The rational function $\frac{P(x)}{Q(x)}$ is **an improper rational function** if $Deg P(x) > Deg Q(x)$. In this case, we can use long division to get

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where Deg $R(x) <$ Deg $Q(x)$.

Example Write

$$\frac{x^3 - 25x + 1}{x^2 - 25}$$

in the form

$$S(x) + \frac{R(x)}{x^2 - 25}$$

where Deg $R(x) < 2$, and evaluate

$$\int \frac{x^3 - 25x + 1}{x^2 - 25}\, dx.$$

Rationalizing Substitutions

Sometimes we can make a substitution which allows us to express an integral as a rational function.

Example

$$\int \frac{\sqrt{x}}{x+1}dx \qquad\qquad \int \frac{\cos\theta}{3\sin^2\theta + 2\sin\theta}d\theta$$