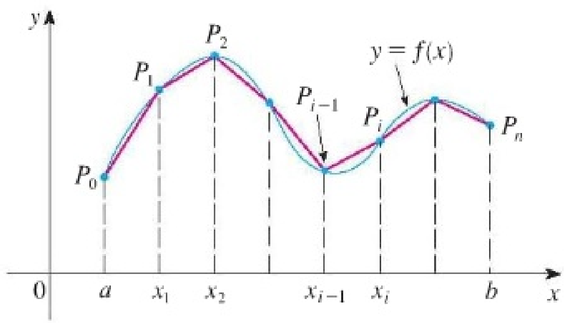
Lecture 16 : Arc Length

In this section, we derive a formula for the length of a curve $y = f(x)$ on an interval $[a, b]$. We will assume that f is continuous and differentiable on the interval $[a, b]$ and we will assume that its derivative f' is also continuous on the interval $[a, b]$. We use Riemann sums to approximate the length of the curve over the interval and then take the limit to get an integral.



We see from the picture above that

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Letting $\Delta x = \frac{b-a}{n} = |x_{i-1} - x_i|$, we get

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \Delta x \sqrt{1 + \Big[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\Big]^2}$$

Now by the mean value theorem from last semester, we have $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f'(x_i^*)$ for some x_i^* in the interval $[x_{i-1}, x_i]$. Therefore

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

giving us

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad L = \int_a^b \sqrt{1 + \Big[\frac{dy}{dx}\Big]^2} dx}$$

Example Find the arc length of the curve $y = \frac{2x^{3/2}}{3}$ from $(1, \frac{2}{3})$ to $(2, \frac{4\sqrt{2}}{3})$.

Example Find the arc length of the curve $y = \frac{e^x + e^{-x}}{2}, \quad 0 \le x \le 2$.

Example Set up the integral which gives the arc length of the curve $y = e^x, \quad 0 \le x \le 2$. Indicate how you would calculate the integral. (the full details of the calculation are included at the end of your lecture).

For a curve with equation $x = g(y)$, where $g(y)$ is continuous and has a continuous derivative on the interval $c \le y \le d$, we can derive a similar formula for the arc length of the curve between $y = c$ and $y = d$.

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2} dy \quad \text{or} \quad L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2} dy}$$

Example Find the length of the curve $24xy = y^4 + 48$ from the point $(\frac{4}{3}, 2)$ to $(\frac{11}{4}, 4)$.

We cannot always find an antiderivative for the integrand to evaluate the arc length. However, we can use Simpson's rule to estimate the arc length.

Example Use Simpson's rule with $n = 10$ to estimate the length of the curve

$$x = y + \sqrt{y}, \quad 2 \le y \le 4$$

$$dx/dy = 1 + \frac{1}{2\sqrt{y}}$$

$$L = \int_2^4 \sqrt{1 + \left[\frac{dx}{dy}\right]^2} \, dy = \int_2^4 \sqrt{1 + \left[1 + \frac{1}{2\sqrt{y}}\right]^2} \, dy = \int_2^4 \sqrt{2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}} \, dy$$

With $n = 10$, Simpson's rule gives us

$$L \approx S_{10} = \frac{\Delta y}{3}\left[g(2)+4g(2.2)+2g(2.4)+4g(2.6)+2g(2.8)+4g(3)+2g(3.2)+4g(3.4)+2g(3.6)+4g(3.8)+g(4)\right]$$

where $g(y) = \sqrt{2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}}$ and $\Delta y = \frac{4-2}{10}$.

y_i	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
y_i	2	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
$g(y_i) = \sqrt{2 + \frac{1}{\sqrt{y}} + \frac{1}{4y}} \approx$	1.68	1.67	1.66	1.65	1.64	1.63	1.62	1.62	1.61	1.61	1.60

We get

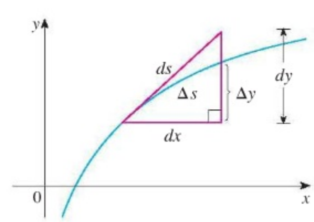
$$S_{10} \approx 3.269185$$

The distance along a curve with equation $y = f(x)$ from a fixed point $(a, f(a))$ is a function of x. It is called the **arc length function** and is given by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}dt.$$

From the fundamental theorem of calculus, we see that $s'(x) = \sqrt{1 + [f'(x)]^2}$. In the language of differentials, this translates to

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \quad \text{or} \quad (ds)^2 = (dx)^2 + (dy)^2$$



Example Find the arc length function for the curve $y = \frac{2x^{3/2}}{3}$ taking $P_0(1, 3/2)$ as the starting point.

Worked Examples

Example Find the length of the curve $y = e^x, \quad 0 \le x \le 2$.

$$L = \int_0^2 \sqrt{1 + \left[\frac{dy}{dx}\right]^2} dx = \int_0^2 \sqrt{1 + \left[e^x\right]^2} dx = \int_0^2 \sqrt{1 + e^{2x}} dx$$

Let $u = e^x$, $du = udx$ or $dx = du/u$. $u(0) = 1$ and $u(2) = e^2$. This gives

$$\int_0^2 \sqrt{1 + e^{2x}} dx = \int_1^{e^2} \frac{\sqrt{1 + u^2}}{u} du$$

Letting $u = \tan\theta$, where $-\pi/2 \le \theta \le \pi/2$, we get $\sqrt{1 + u^2} = \sqrt{1 + \tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$ and $du = \sec^2\theta d\theta$

$$\int_{\frac{\pi}{4}}^{\tan^{-1}(e^2)} \frac{\sec\theta}{\tan\theta} \sec^2\theta d\theta$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(e^2)} \frac{\sec^3\theta}{\tan\theta} d\theta = \int_{\frac{\pi}{4}}^{\tan^{-1}(e^2)} \frac{\sec^3\theta\tan\theta}{\tan^2\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(e^2)} \frac{\sec^3\theta\tan\theta}{\sec^2\theta - 1} d\theta$$

Letting $w = \sec\theta$, we have $w(\frac{\pi}{4}) = \sqrt{2}$, $w(\tan^{-1}(e^2)) = \sqrt{1 + e^4}$ from a triangle and $dw = \sec\theta\tan\theta$. Our integral becomes

$$\int_{\sqrt{2}}^{\sqrt{1+e^4}} \frac{w^2}{w^2 - 1} dw = \int_{\sqrt{2}}^{\sqrt{1+e^4}} 1 + \frac{1}{w^2 - 1} dw = \int_{\sqrt{2}}^{\sqrt{1+e^4}} 1 + \frac{1/2}{w - 1} - \frac{1/2}{w + 1} dw$$

$$= w + \frac{1}{2}\ln|w - 1| - \frac{1}{2}\ln|w + 1| \Big]_{\sqrt{2}}^{\sqrt{1+e^4}} = w + \frac{1}{2}\ln\left|\frac{w - 1}{w + 1}\right| \Big]_{\sqrt{2}}^{\sqrt{1+e^4}}$$

$$= \sqrt{1 + e^4} - \sqrt{2} + \frac{1}{2}\ln\left|\frac{\sqrt{1 + e^4} - 1}{\sqrt{1 + e^4} + 1}\right| - \frac{1}{2}\left|\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right|.$$