Lecture 17 : Moments and Centers of Mass

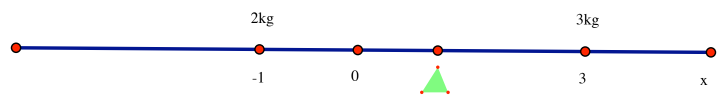
If we have masses $m_1, m_2, \ldots, m_n$ at points $x_1, x_2, \ldots, x_n$ along the x-axis, the **moment of the system around the origin** is

$$\boxed{M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n}.$$

The **center of mass of the system** is

$$\boxed{\bar{x} = \frac{M_0}{m}, \quad \text{where} \;\; m = m_1 + m_2 + \cdots + m_n.}$$

Example We have a mass of 3 kg at a distance 3 units to the right the origin and a mass of 2 kg at a distance of 1 unit to the left of the origin on the rod below, find the moment about the origin. Find the center of mass of the system.



By a **Law of Archimedes** if we have masses m_1 and m_2 on a rod (of negligible mass) on opposite sides of a fulcrum, at distances d_1 and d_2 from the fulcrum, the rod will balance if $m_1d_1 = m_2d_2$. (or in general if we place masses $m_1, m_2, \ldots, m_n$ at distances $d_1, d_2, \ldots d_n$ from the fulcrum the rod will balance if the center of mass is at the fulcrum.)

If we place a fulcrum at the center of mass of the rod above, we see that the rod will balance:
Verify that

$$m_1d_1 = m_2d_2 \qquad \text{or} \qquad 3|\bar{x} - 3| = 2|\bar{x} - (-1)|.$$

Note that **a system with all of the mass placed at the center of mass, has the same moment as the original system.**
Check this for the system above. (That is, check that a system with a mass of 5 kg placed at $\bar{x} = \frac{7}{5}$ has the same moment and center of mass)

For a two dimensional system, we use x and y axes for reference. We now have moments about each axis. If we have a system with masses $m_1, m_2, \ldots, m_n$ at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ respectively, then the moment about the y axis is given by

$$\boxed{M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n}$$

and the moment about the x axis is given by

$$\boxed{M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n.}$$

These moments measure the tendency of the system to rotate about the x and y axes respectively. The **Center of Mass** of the system is given by $(\bar{x}, \bar{y})$ where

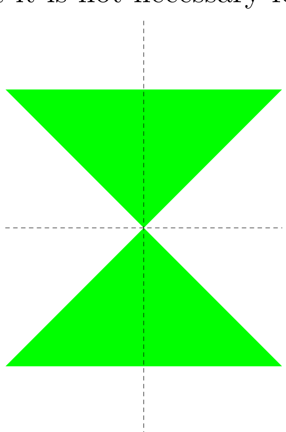
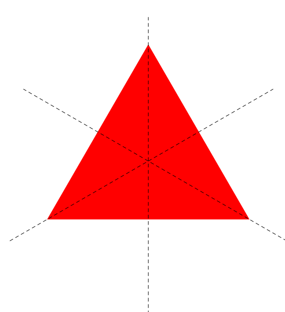
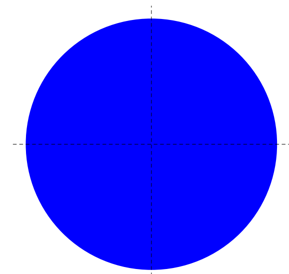
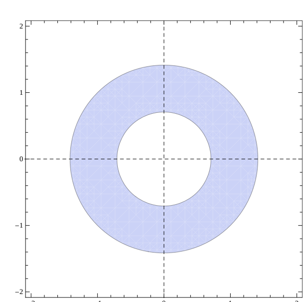
$$\boxed{\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} \quad \text{for} \quad m = m_1 + m_2 + \cdots + m_n.}$$

Example Find the moments and center of mass of a system of objects that have masses

kg	2	1	6
position	$(7,1)$	$(0,0)$	$(-3,0)$

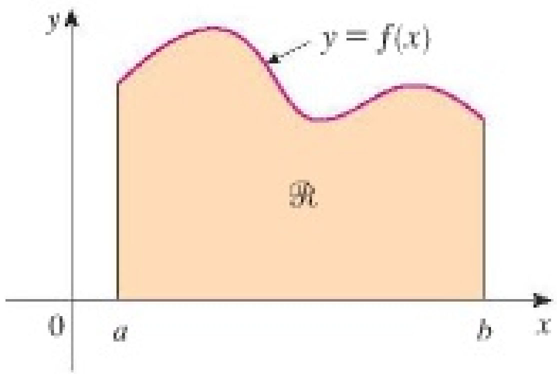
Note that **a system with all of the mass placed at the center of mass, has the same moments as the original system.** Check this for the system above.

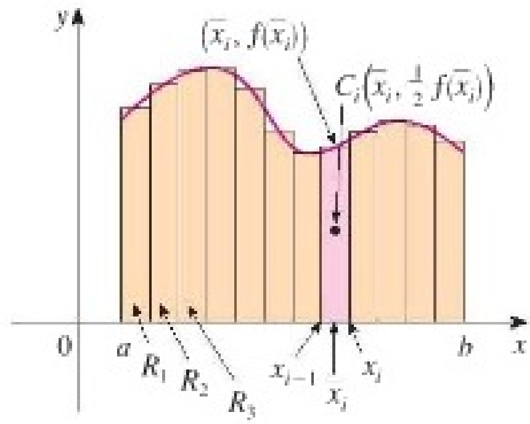
If we have a a thin plate (which occupies a region $\mathfrak{R}$ of the plane) with uniform density ρ, we are interested in calculating its moments about the x and y axis (M_x and M_y respectively) and its center of mass or **centroid**. These are defined in a way that agrees with our intuition. A line of symmetry of the plate (or region) is a line for which a 180^o rotation of the plate about the line makes no change in the plate's appearance. **The Centroid or center of mass lies on each line of symmetry of the plate**. Hence if we have 2 different lines of symmetry, the centroid must be at their intersection. (Note that it is not necessary for the center of mass to lie in the planar region).

The **moments** of the plate or region are defined so that if all of the mass of the plate is centered at the centroid, the system has the same moments. Also the moments of the union of two non-overlapping regions should be the sum of the moments of the individual regions.

Naturally :) we use Riemann sums to calculate the moments of a region (or a plate shaped like that region with constant density) and we will start with a region, $\mathfrak{R}$, under a curve $y = f(x) \geq 0$ for $a \leq x \leq b$.





In order to estimate the moments for such a region (with constant density ρ), we divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, $0 \leq i \leq n$, each of length $\Delta x = \frac{b-a}{n}$. We approximate the moment of the region beneath the curve above the subinterval $[x_{i-1}, x_i]$ by the moment of a rectangle above the subinterval $[x_{i-1}, x_i]$ of height $f(\frac{x_{i-1}+x_i}{2})$. The centroid of this rectangle is $C_i(\bar{x}_i, \frac{1}{2}f(\bar{x}_i))$ where $\bar{x}_i = \frac{x_{i-1}+x_i}{2}$. The area of this approximating rectangle is $\Delta x f(\bar{x}_i)$ and hence its mass is $\rho \Delta x f(\bar{x}_i)$

For the rectangle R_i we calculate its moment about the y-axis as if it were a system with all of its mass at its center of mass $C_i(\bar{x}_i, \frac{1}{2}f(\bar{x}_i))$. Hence the moment of this rectangle about the y-axis is

$$M_y(R_i) = \text{ mass} \times \bar{x}_i = \rho \Delta x f(\bar{x}_i)\bar{x}_i = \rho \bar{x}_i \Delta x f(\bar{x}_i).$$

We add the moments for the approximating rectangles to get an approximation of the moment of the entire region $\mathfrak{R}$;

$$M_y \approx \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x.$$

Taking limits we get the moment of the region $\mathfrak{R}$ to be

$$\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x = \rho \int_a^b x f(x) dx.}$$

Similarly we find the moment about the x-axis for the approximating rectangle R_i to be

$$M_x(R_i) = \text{ mass} \times \bar{y}_i = \rho \Delta x f(\bar{x}_i) \frac{1}{2} f(\bar{x}_i) = \frac{\rho}{2}[f(\bar{x}_i)]^2 \Delta x.$$

and the moment about the x-axis of the region $\mathfrak{R}$ is given by

$$\boxed{M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\rho}{2}[f(\bar{x}_i)]^2 \Delta x = \frac{\rho}{2} \int_a^b [f(x)]^2 dx.}$$

The **Center of Mass or centroid** of the region $\mathfrak{R}$ is given by $(\bar{x}, \bar{y})$ where $m\bar{x} = M_y$ and $m\bar{y} = M_x$ where m is the mass of the entire region $\mathfrak{R}$.

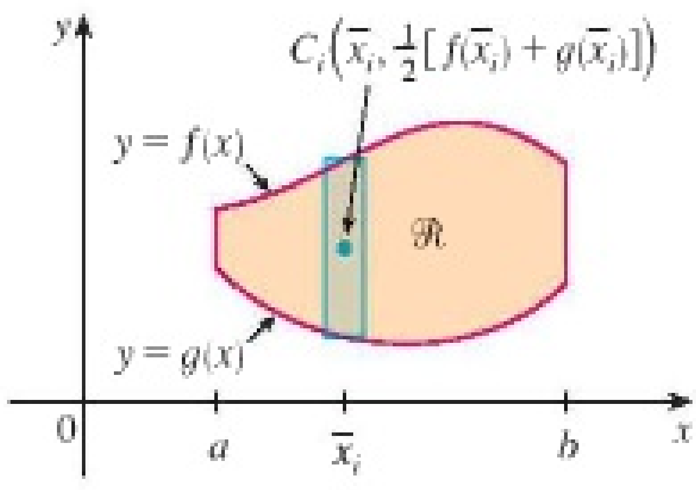
$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x) dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}}$$

and

$$\boxed{\bar{y} = \frac{M_x}{m} = \frac{\frac{\rho}{2} \int_a^b [f(x)]^2 dx}{\rho \int_a^b f(x) dx} = \frac{\frac{1}{2} \int_a^b [f(x)]^2 dx}{\int_a^b f(x) dx}}$$

Note that the density has no effect on the center of mass.

Example Find the centroid of the region bounded by the curve $y = \frac{1}{x^2}$, the x-axis, the line $x = 1$ and the line $x = 2$.



If the region $\mathfrak{R}$ is bounded above by the curve $y = f(x)$ and below by the curve $y = g(x)$, where $f(x) \geq g(x) \geq 0$, we have the moments of a plate with that shape and constant density ρ are given by

$$\boxed{M_y = \rho \int_a^b x[f(x) - g(x)]dx \quad \text{and} \quad M_x = \frac{\rho}{2} \int_a^b [f(x)]^2 - [g(x)]^2 dx.}$$

and the centroid of the region $\mathfrak{R}$ is given by

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]dx \qquad \bar{y} = \frac{1}{2A}\int_a^b [f(x)]^2 - [g(x)]^2 dx}$$

where A denotes the area of the region $\mathfrak{R}$,

$$A = \int_a^b f(x) - g(x)dx.$$

Example Find the centroid of the region bounded above by $y = x + 2$ and below by the curve $y = x^2$.

Extras

Example Find the moment M_0 and center of mass of a system, consisting of a rod with negligible weight, with a mass of 2 kg placed 3 units to the right of the origin, a mass of 6 kg placed 5 units to the left of the origin, a mass of 10 kg placed 8 units to the right of the origin and a mass of 10 kg placed at the origin.

$$M_0 = m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 6(-5) + 10(0) + 2(3) + 10(8) = -30 + 6 + 80 = 56.$$

$$\bar{x} = \frac{M_0}{m} = \frac{56}{m_1 + m_2 + m_3 + m_4} = \frac{56}{28} = 2.$$

Example Find the centroid of the region bounded by the curve $y = \frac{1}{x^2}$, the x-axis, the line $x = 1$ and the line $x = 2$.

Covered in class

Find the moment about the x-axis of a plate with shape described as in the previous example and density $\rho = 1/2$ kg per unit area. Find the moment about the y-axis of such a plate.

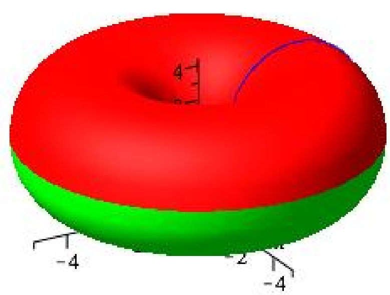
Using our calculations from above we get

$$M_x = \frac{\rho}{2}\int_1^2 [f(x)]^2 dx = \frac{1}{2}\left[\frac{7}{48}\right].$$

$$M_y = \rho\int_1^2 xf(x)dx = \frac{1}{2}\ln 2.$$

Theorem of Pappus Ler $\mathfrak{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathfrak{R}$ is rotated around the line l, the **volume of the resulting solid** is the product of $A =$ the area of $\mathfrak{R}$ and the distance travelled by the centroid of $\mathfrak{R}$, $(\bar{x}, \bar{y})$.

Example Find the volume of the solid generated by rotating the region between the curve $y = 2 + \sqrt{4 - (x-3)^2}$ and the curve $y = 2 - \sqrt{4 - (x-3)^2}$ about the y axis.
The solid looks like the doughnut shown below:



The region between the above curves is a circle of radius 2 satisfying the equation

$$(x-3)^2 + (y-2)^2 = 4.$$

It is a circle of radius 2 centered at the point $(3, 2)$. The centroid of that region is on the lines of symmetry of the region which meet at the center $(\bar{x}, \bar{y}) = (3, 2)$. The area of the region is 4π and the distance travelled by the centroid is $2\pi(3)$ since the centroid is 3 units from the y axis. Hence by Pappus, the volume of the doughnut is

$$2\pi(3) \times 4\pi = 24\pi^2.$$