Lecture 2 : The Natural Logarithm.

Recall
\[
\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1.
\]

What happens if \( n = -1 \)?

**Definition** We can define a function which is an anti-derivative for \( x^{-1} \) using the Fundamental Theorem of Calculus: We let
\[
\ln x = \int_1^x \frac{1}{t} \, dt, \quad x > 0.
\]
This function is called the natural logarithm.

**Note** that \( \ln(x) \) is the area under the continuous curve \( y = \frac{1}{t} \) between 1 and \( x \) if \( x > 1 \) and minus the area under the continuous curve \( y = \frac{1}{t} \) between 1 and \( x \) if \( x < 1 \).

We have \( \ln(2) \) is the area of the region shown in the picture on the left above and \( \ln(1/2) \) is minus the area of the region shown in the picture on the right above.

I do not have a formula for \( \ln(x) \) in terms of functions studied before, however I could estimate the value of \( \ln(2) \) using a Riemann sum. The approximating rectangles for a left Riemann sum with 10 approximating rectangles is shown below. Their area adds to 0.718771 (to 6 decimal places). If we took the limit of such sums as the number of approximating rectangles tends to infinity, we would get the actual value of \( \ln(2) \), which is 0.693147 (to 6 decimal places). The natural logarithm function is a built-in function on most scientific calculators.

With very little work, using a right Riemann sum with 1 approximating rectangle, we can get a lower bound for \( \ln(2) \). The picture below demonstrates that
\[
\ln 2 = \int_1^2 \frac{1}{t} \, dt > 1/2.
\]
Properties of the Natural Logarithm:

We can use our tools from Calculus I to derive a lot of information about the natural logarithm.

1. Domain = \((0, \infty)\) (by definition)

2. Range = \((-\infty, \infty)\) (see later)

3. \(\ln x > 0\) if \(x > 1\), \(\ln x = 0\) if \(x = 1\), \(\ln x < 0\) if \(x < 1\).

   This follows from our comments above after the definition about how \(\ln(x)\) relates to the area under the curve \(y = 1/x\) between 1 and \(x\).

4. \(\frac{d(\ln x)}{dx} = \frac{1}{x}\)

   This follows from the definition and the Fundamental Theorem of Calculus.

5. The graph of \(y = \ln x\) is increasing, continuous and concave down on the interval \((0, \infty)\).

   Let \(f(x) = \ln(x)\), \(f'(x) = 1/x\) which is always positive for \(x > 0\) (the domain of \(f\)). Therefore the graph of \(f(x)\) is increasing on its domain. We have \(f''(x) = -\frac{1}{x^2}\) which is always negative, showing that the graph of \(f(x)\) is concave down. The function \(f\) is continuous since it is differentiable.

6. The function \(f(x) = \ln x\) is a one-to-one function.

   Since \(f'(x) = 1/x\) which is positive on the domain of \(f\), we can conclude that \(f\) is a one-to-one function.

7. Since \(f(x) = \ln x\) is a one-to-one function, there is a unique number, \(e\), with the property that

   \[
   \ln e = 1.
   \]

   We have \(\ln(1) = 0\) since \(\int_1^1 1/t \, dt = 0\). Using a Riemann sum with 3 approximating rectangles, we see that \(\ln(4) > 1/1 + 1/2 + 1/3 > 1\). Therefore by the intermediate value theorem, since \(f(x) = \ln(x)\) is continuous, there must be some number \(e\) with \(1 < e < 4\) for which \(\ln(e) = 1\). This number is unique since the function \(f(x) = \ln(x)\) is one-to-one.
We will be able to estimate the value of $e$ in the next section with a limit. $e \approx 2.7182818284590$.

The following properties are very useful when calculating with the natural logarithm:

\[
\begin{align*}
(i) \quad & \ln 1 = 0 \\
(ii) \quad & \ln(ab) = \ln a + \ln b \\
(iii) \quad & \ln \left(\frac{a}{b}\right) = \ln a - \ln b \\
(iv) \quad & \ln a^r = r \ln a
\end{align*}
\]

where $a$ and $b$ are positive numbers and $r$ is a rational number.

**Proof** (ii) We show that $\ln(ax) = \ln a + \ln x$ for a constant $a > 0$ and any value of $x > 0$. The rule follows with $x = b$. Let $f(x) = \ln x$, $x > 0$ and $g(x) = \ln(ax)$, $x > 0$. We have $f'(x) = \frac{1}{x}$ and $g'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$.

Since both functions have equal derivatives, $f(x) + C = g(x)$ for some constant $C$. Substituting $x = 1$ in this equation, we get $\ln 1 + C = \ln a$, giving us $C = \ln a$ and $\ln ax = \ln a + \ln x$.

(iii) Note that $0 = \ln 1 = \ln \frac{a}{a} = \ln a \cdot \frac{1}{a} = \ln a + \ln \frac{1}{a}$, giving us that $\ln \frac{1}{a} = -\ln a$.

Thus we get $\ln \frac{a}{b} = \ln a + \ln \frac{1}{b} = \ln a - \ln b$.

(iv) Comparing derivatives, we see that

\[
\frac{d(\ln x^r)}{dx} = \frac{rx^{r-1}}{x^r} = \frac{r}{x} = \frac{d(r \ln x)}{dx}.
\]

Hence $\ln x^r = r \ln x + C$ for any $x > 0$ and any rational number $r$. Letting $x = 1$ we get $C = 0$ and the result holds.

**Example** Expand

\[
\ln \frac{x^2 \sqrt{x^2 + 1}}{x^3}
\]

using the rules of logarithms.

**Example** Express as a single logarithm:

\[
\ln x + 3 \ln(x + 1) - \frac{1}{2} \ln(x + 1).
\]
**Example** Evaluate $\int_1^{e^2} \frac{1}{t} dt$

We can use the rules of logarithms given above to derive the following information about limits.

We have:

$$\lim_{x \to \infty} \ln x = \infty, \quad \lim_{x \to 0} \ln x = -\infty.$$

**Proof** We saw above that $\ln 2 > 1/2$. If $x > 2^n$, then $\ln x > \ln 2^n$ (Why?). So $\ln x > n \ln 2 > n/2$. Hence as $x \to \infty$, the values of $\ln x$ also approach $\infty$.

Also $\ln \frac{1}{2^n} = -n \ln 2 < -n/2$. Thus as $x$ approaches 0 the values of $\ln x$ approach $-\infty$.

Note that we can now draw a reasonable sketch of the graph of $y = \ln(x)$, using all of the information derived above.

**Example** Find the limit $\lim_{x \to \infty} \ln(\frac{1}{x+1})$.

We can extend the applications of the natural logarithm function by composing it with the absolute value function. We have:

$$\ln |x| = \begin{cases} 
\ln x & x > 0 \\
\ln(-x) & x < 0 
\end{cases}$$

This is an even function with graph
We have $\ln|x|$ is also an antiderivative of $1/x$ with a larger domain than $\ln(x)$.

\[
\frac{d}{dx} (\ln |x|) = \frac{1}{x} \quad \text{and} \quad \int \frac{1}{x} \, dx = \ln |x| + C
\]

We can use the chain rule and integration by substitution to get

\[
\frac{d}{dx} (\ln |g(x)|) = \frac{g'(x)}{g(x)} \quad \text{and} \quad \int \frac{g'(x)}{g(x)} \, dx = \ln |g(x)| + C
\]

**Example** Differentiate $\ln|\sqrt{x - 1}|$.

**Example** Find the integral

\[
\int \frac{x}{3 - x^2} \, dx.
\]

**Logarithmic Differentiation**

To differentiate $y = f(x)$, it is often easier to use logarithmic differentiation:

1. Take the natural logarithm of both sides to get $\ln y = \ln(f(x))$.
2. Differentiate with respect to $x$ to get $\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \ln(f(x))$
3. We get $\frac{dy}{dx} = y \frac{d}{dx} \ln(f(x)) = f(x) \frac{d}{dx} \ln(f(x))$.

**Example** Find the derivative of $y = \sqrt[4]{x^2 + 1}$. 

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Extra Examples

Please try to work through these questions before looking at the solutions.

Example Expand \( \ln\left(\frac{e^{\sqrt{x^2+1}}}{\sqrt[3]{a^2+1}}\right) \)

Example Differentiate \( \ln|\sqrt[3]{x} - 1| \).

Example Find \( d/dx \ln(|\cos x|) \).

Example Find the integral
\[
\int \cot x \, dx
\]

Example Find the integral
\[
\int e^x \frac{1}{x \ln x} \, dx.
\]

Example Find the derivative of \( y = \frac{\sin^2 x \tan^4 x}{(x^2-1)^2} \).

Old Exam Question Differentiate the function
\[
f(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.
\]
Solutions

**Example** Expand \( \ln(\frac{e^{2\sqrt{a^2+1}}}{b^3}) \)

\[
\ln(\frac{e^{2\sqrt{a^2+1}}}{b^3}) = \ln(e^{2\sqrt{a^2+1}}) - \ln(b^3) = \ln(e^2) + \ln(\sqrt{a^2+1}) - 3 \ln b
\]

\[
= 2 \ln e + \frac{1}{2} \ln(a^2+1) - 3 \ln b = 2 + \frac{1}{2} \ln(a^2+1) - 3 \ln b.
\]

**Example** Differentiate \( \ln(\sqrt[x]{x-1}) \).

We use the chain rule here

\[
\frac{d}{dx} \ln(\sqrt[x]{x-1}) = \frac{1}{\sqrt[x]{x-1}} \cdot \frac{1}{3} (x-1)^{-2/3} = \frac{1}{3(x-1)}.
\]

**Example** Find \( d/dx \ln(|\cos x|) \).

Again, we use the chain rule

\[
\frac{d}{dx} \ln(|\cos x|) = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x.
\]

**Example** Find the integral \( \int \cot x \, dx \)

\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx.
\]

We use substitution. Let \( u = \sin x \), \( du = \cos x \, dx \).

\[
\int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln |\sin x| + C.
\]

**Example** Find the integral \( \int e^2 \frac{1}{x \ln x} \, dx \).

We use substitution. Let \( u = \ln x \), \( du = \frac{1}{x} \, dx \). \( u(e) = \ln e = 1 \), \( u(e^2) = \ln e^2 = 2 \).

\[
\int_e^{e^2} \frac{1}{x \ln x} \, dx = \int_1^2 \frac{du}{u} = \ln |u|_1^2 = \ln 2 - \ln 1 = \ln 2.
\]
Example  Find the derivative of \( y = \frac{\sin^2 x \tan^4 x}{(x^2 - 1)^2} \).

We use Logarithmic differentiation. If \( y = \frac{\sin^2 x \tan^4 x}{(x^2 - 1)^2} \), then

\[
\ln y = \ln(\sin^2 x) + \ln(\tan^4 x) - \ln((x^2 - 1)^2) = 2 \ln(\sin x) + 4 \ln(\tan x) - 2 \ln(x^2 - 1).
\]

Differentiating both sides with respect to \( x \), we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{2 \cos x}{\sin x} + \frac{4 \sec^2 x}{\tan x} - \frac{2(2x)}{x^2 - 1}.
\]

Multiplying both sides by \( y \) and converting to a function of \( x \), we get

\[
\frac{dy}{dx} = y \left[ \frac{2 \cos x}{\sin x} + \frac{4 \sec^2 x}{\tan x} - \frac{4x}{x^2 - 1} \right] = \left( \frac{\sin^2 x \tan^4 x}{(x^2 - 1)^2} \right) \left[ \frac{2 \cos x}{\sin x} + \frac{4 \sec^2 x}{\tan x} - \frac{4x}{x^2 - 1} \right].
\]

Old Exam Question  Differentiate the function

\[
f(x) = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}}.
\]

We use Logarithmic differentiation. If \( y = \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}} \), then

\[
\ln y = 4 \ln(x^2 - 1) - \frac{1}{2} \ln(x^2 + 1).
\]

Differentiating both sides with respect to \( x \), we get

\[
\frac{1}{y} \frac{dy}{dx} = \frac{4(2x)}{x^2 - 1} - \frac{2x}{2(x^2 + 1)}.
\]

Multiplying both sides by \( y \) and converting to a function of \( x \), we get

\[
\frac{dy}{dx} = y \left[ \frac{8x}{x^2 - 1} - \frac{x}{(x^2 + 1)} \right] = \left( \frac{(x^2 - 1)^4}{\sqrt{x^2 + 1}} \right) \left[ \frac{8x}{x^2 - 1} - \frac{x}{(x^2 + 1)} \right].
\]