Lecture 20 : Linear Differential Equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.

The above form of the equation is called the **Standard Form** of the equation.

Example Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y.$$

To solve an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we multiply by a function of of x called an **Integrating Factor**. This function is

$$\boxed{I(x) = e^{\int P(x)dx}.}$$

(we use a particular antiderivative of $P(x)$ in this equation.)
$I(x)$ has the property that

$$\frac{dI(x)}{dx} = P(x)I(x).$$

Multiplying across by $I(x)$, we get an equation of the form

$$I(x)y' + I(x)P(x)y = I(x)Q(x).$$

The left hand side of the above equation is the derivative of the product $I(x)y$. Therefore we can rewrite our equation as

$$\frac{d[I(x)y]}{dx} = I(x)Q(x).$$

Integrating both sides with respect to x, we get

$$\int \frac{d[I(x)y]}{dx}dx = \int I(x)Q(x)dx$$

or

$$I(x)y = \int I(x)Q(x)dx + C$$

giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

(we amalgamate constants in this equation.)

Example Solve the differential equation

$$x\frac{dy}{dx} = x^2 + 3y.$$

Example Solve the initial value problem

$$y' + xy = x, \quad y(0) = -6.$$