Lecture 20/21 : First and Second Order Linear Differential Equations

First Order Linear Differential Equations

A **First Order Linear Differential Equation** is a first order differential equation which can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x), Q(x)$ are continuous functions of x on a given interval.

The above form of the equation is called the **Standard Form** of the equation.

Example Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y.$$

To solve an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

we multiply by a function of of x called an **Integrating Factor**. This function is

$$\boxed{I(x) = e^{\int P(x)dx}.}$$

(we use a particular antiderivative of $P(x)$ in this equation.)
$I(x)$ has the property that

$$\frac{dI(x)}{dx} = P(x)I(x).$$

Multiplying across by $I(x)$, we get an equation of the form

$$I(x)y' + I(x)P(x)y = I(x)Q(x).$$

The left hand side of the above equation is the derivative of the product $I(x)y$. Therefore we can rewrite our equation as

$$\frac{d[I(x)y]}{dx} = I(x)Q(x).$$

Integrating both sides with respect to x, we get

$$\int \frac{d[I(x)y]}{dx}dx = \int I(x)Q(x)dx$$

or

$$I(x)y = \int I(x)Q(x)dx + C$$

giving us a solution of the form

$$\boxed{y = \frac{\int I(x)Q(x)dx + C}{I(x)}}$$

(we amalgamate constants in this equation.)

Example Solve the differential equation

$$x\frac{dy}{dx} = x^2 + 3y.$$

Example Solve the initial value problem

$$y' + xy = x, \quad y(0) = -6.$$

Second Order Linear Differential Equations (Section 17.1) - see e-book

A **Second Order Linear Differential Equation** is a second order differential equation which can be put in the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where $P(x), Q(x), R(x)$ and $G(x)$ are continuous functions of x on a given interval.

If $G = 0$ then the equation is called **homogeneous**. Otherwise is called **nonhomogeneous**. In this lecture, we will solve homogeneous second order linear equations, in the next lecture, we will cover nonhomogeneous second order linear equations. In general is very difficult to solve second order linear equations, general ones will be solved in a differential equations class. Here we will solve homogeneous second order linear equations with constant coefficients, i.e. equations of the type:

$$ay'' + by' + cy = 0,$$

where a, b and c are constants and $a \neq 0$.

Example $y'' + 2y' - 8y + 0$ is a homogeneous second order linear equation with constant coefficients.

Theorem If $y_1(x)$ and $y_2(x)$ are solutions to the differential equation $ay'' + by' + cy = 0$, then every function of the form $c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are constants, is also a solution of the equation.

Proof $ay'' + by' + cy = c_1[ay_1'' + by_1' + cy_1] + c_2[ay_2'' + by_2' + cy_2] = 0 + 0 = 0.$

Theorem Let $y_1(x)$ and $y_2(x)$ are (non-zero) solutions to the differential equation $ay''+by'+cy = 0$, where $y_1 \neq cy_2$ (equivalently $y_2 \neq ky_1$), for constants c and k. Then the general solution to the differential equation $ay'' + by' + cy = 0$ is given by

$$c_1y_1(x) + c_2y_2(x).$$

Note: If $y_1 \neq cy_2$ for any real number $c \neq 0$, we say y_1 and y_2 are **linearly independent** solutions.

Example: Harmonic motion Check that $y_1(x) = \sin(\sqrt{2}x)$ and $y_2(x) = \cos(\sqrt{2}x)$ are solutions to the differential equation

$$\frac{d^2y}{dx^2} + 2y = 0.$$

Since $\sin(2x) \neq c\cos(2x)$, we have the general solution to this differential equation is given by

$$c_1\sin(2x) + c_2\cos(2x).$$

Auxiliary Equation/Characteristic equation To solve a second order differential equation of the form

$$ay'' + by' + cy = 0, \qquad a \neq 0,$$

we must consider the roots of **auxiliary equation (or characteristic equation)** given by $ar^2+br+c = 0$.

Example What are the roots of the auxiliary equation of the differential equation $y'' + 2y' - 8y = 0$?

Consider a homogeneous second order linear equation with constant coefficients

$$ay'' + by + c = 0.$$

For this type of equation, it is reasonable to expect solutions of the form $y = e^{rx}$. Substituting such a function into the equation and solving for r(see your book for details), we find that r is a root of the auxiliary equation. Recall the roots of the auxiliary equation are given by

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We demonstrate how to extract 2 linearly independent solutions from the auxiliary equation in each of 3 possible cases below. The results depend on the sign of the **discriminant** $b^2 - 4ac$.

Method for solving : Given a homogeneous second order linear equation with constant coefficients

$$ay'' + by + c = 0, \quad a \neq 0$$

Step 1: Write down the auxiliary equation $ar^2 + br + c$. Calculate the discriminant $b^2 - 4ac$.

Step 2: Calculate r_1 and r_2 above.

Step 3: If $b^2 - 4ac > 0$, ($r_1 \neq r_2$, both real) In this case, $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent solutions to the differential equation and the general solution is given by

$$c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

If $b^2 - 4ac = 0$, ($r = r_1 = r_2$, real). In this case $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent solutions to the differential equation (check book for details) and the general solution is given by

$$c_1 e^{rx} + c_2 x e^{rx}.$$

If $b^2 - 4ac < 0$, (Complex roots, $r_1 = \alpha + i\beta, r_2 = \alpha + i\beta, \quad \alpha, \beta \in \mathbb{R}, \quad i = \sqrt{-1}.$). By definition, $e^{\alpha+i\beta} = e^{\alpha}(cos\beta + i \sin\beta)$. As before, $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions to the differential equation. We can rearrange the general solution $C_1 e^{r_1 x} + C_2 e^{r_2 x}$ (see book for details) to show that all solutions are of the form :

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad c_1, c_2 \text{real or complex}$$

Solutions of $ay'' + by' + cy = 0$:

Roots of $ar^2 + br + c = 0$	General Solution
r_1, r_2 real and distinct	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$r_1 = r_2 = r$	$y = c_1 e^{rx} + c_2 x e^{rx}$
r_1, r_2 complex : $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$

Example What is the general solution to the differential equation $y'' + 2y' - 8y = 0$?

Example Solve the differential equation $y'' - 6y' + 9y = 0$.

Example Solve the differential equation $y'' + 3y = 0$.

Example Solve the differential equation $y'' + y' + 3y = 0$.

Initial value problems

An Initial Value Problem for a second-order differential equations asks for a specific solution to the differential equation that also satisfies **TWO initial conditions** of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

Note: for a differential equation of type $ay'' + by' + cy = G(x), \quad a \neq 0$, a solution to an initial value problem always exists and is unique. You will see a proof of this in later courses on differential equations.

Example Solve the initial value problem

$$2y'' + 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

[Note: here $y(t)$ might denote the position of an object moving along the y axis after t minutes. The object starts at a position of 1 meter above the origin with an initial velocity of 2 meters per minute and its acceleration at time t is given by $y'' = (-3y' - y)/2$]

Boundary Value Problems

A **boundary value problem** for a second-order differential equations asks for a specific solution to the differential equation that also satisfies **TWO conditions** of the form

$$y(x_0) = y_0, \quad y(x_1) = y_1.$$

Note Such a boundary value problem does not always have a solution.

Example Find a solution to the boundary value problem,

$$y'' + 3y = 0, \quad y(0) = 3, \quad y\left(\frac{\pi}{2\sqrt{3}}\right) = 5.$$