Lecture 22 : NonHomogeneous Linear Equations (Section 17.2)

The solution of a second order **nonhomogeneous** linear differential equation of the form

$$ay'' + by' + cy = G(x)$$

where $a \neq 0$ and $G(x)$ are continuous functions of x on a given interval is of the form

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of $ay'' + by' + cy = G(x)$ and $y_c(x)$ is the general solution of the **complementary equation/ corresponding homogeneous equation** $ay'' + by' + cy = 0$.

Since we already know how to find the general solution to the corresponding homogeneous equation, we need a method to find a particular solution to the equation. One such methods is described below. This method may not always work. A second method which is always applicable is demonstrated in the extra examples.

The method of Undetermined Coefficients (for particular solution to $ay'' + by' + cy = G(x)$)

If $G(x)$ is a polynomial it is reasonable to guess that there is a particular solution, $y_p(x)$ which is a polynomial in x of the same degree as $G(x)$ (because if y is such a polynomial, then $ay'' + by' + c$ is also a polynomial of the same degree.)
Method to find a particular solution: Substitute $y_p(x) =$ a polynomial of the same degree as G into the differential equation and determine the coefficients.

Example Solve the differential equation:

$$y'' + 3y' + 2y = x^2.$$

If $G(x)$ is of the form Ce^{kx}, where C and k are constants, then we use a trial solution of the form $y_p(x) = Ae^{kx}$ and solve for A if possible.

Example Solve $y'' + 9y = e^{-4x}$.

If $G(x)$ is of the form $C \cos kx$ or $C \sin kx$, where C and k are constants, then we use a trial solution of the form $y_p(x) = A\cos(kx) + B\sin(kx)$ and solve for A and B if possible.
Here we use the fact that if $K_1 \cos(\alpha x) + K_2 \sin(\alpha x) = 0$ for constants K_1, K_2, α, where $\alpha \neq 0$, Then we must have $K_1 = K_2 = 0$.

Example Solve $y'' - 4y' - 5y = \cos(2x)$.

Troubleshooting If the trial solution y_p is a solution of the corresponding homogeneous equation, then it cannot be a solution to the non-homogeneous equation. In this case, we multiply the trial solution by x (or x^2 or x^3 ... as necessary) to get a new trial solution that does not satisfy the corresponding homogeneous equation. Then proceed as above.

Example Solve $y'' - y = e^x$.

To solve the equation $ay'' + by' + cy = G_1(x) + G_2(x)$, we can find particular solutions, y_{p1} and y_{p2} to of the equations

$$ay'' + by' + cy = G_1(x), \qquad ay'' + by' + cy = G_2(x)$$

separately. The general solution of the equation $ay'' + by' + cy = G_1(x) + G_2(x)$ is $y_{p1} + y_{p2} + y_c$, where y_c is the solution of the corresponding homogeneous equation $ay'' + by' + cy = 0$.

Example For the equation $y'' + y' + y = x^2 + e^x$, we use a trial solution of the form

$$y_p = (Ax^2 + Bx + C) + De^x.$$

Extras

If $G(x)$ is a product of functions of the previous type, then we take the trial solution to be a product of functions of the appropriate type for each component of the product.
Example For the equation $y'' + y' + y = x^2 e^x \cos(2x)$, the trial solution would be of the form

$$y_p = (Ax^2 + Bx + C)e^x \cos(2x) + (Dx^2 + Ex + F)e^x \sin(2x).$$

Note we can lump constants together.
For the Equation $y'' + y' + y = xe^x$, the trial solution would be of the form

$$y_p = (Ax + B)e^x$$

Method of Variation of Parameters (for particular solution to $ay'' + by' + cy = G(x)$)

1. Find the solution to the corresponding homogeneous equation $ay'' + by' + cy = 0$, which is of the form:

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x).$$

2. We look for a particular solution of the equation $ay'' + by' + cy = G(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

(making the parameters c_1 and c_2 functions of x).
3. We calculate y_p' to get

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2').$$

4. We impose the condition

$$u_1'y_1 + u_2'y_2 = 0$$

Note: now $y_p' = u_1y_1' + u_2y_2'$ and is easier to differentiate.
5. Evaluate y_p'' and substitute for y_p'', y_p' and y_p in the equation

$$ay_p'' + by_p' + cy_p = G(x).$$

This equation simplifies to an equation of the form $a(u_1'y_1' + u_2'y_2') = G$.
6. Solve for u_1' and u_2' using both equations

$$u_1'y_1 + u_2'y_2 = 0, \quad a(u_1'y_1' + u_2'y_2') = G.$$

7. Integrate the resulting u_1' and u_2' to get functions u_1 and u_2.
Now you can describe y_p and hence $y = y_p + y_c$.

Example Solve the differential equation $y'' + 12y' + 32y = \sin(e^{4x})$.

1. Find the solution to the corresponding homogeneous equation $y'' + 12y' + 32y = 0$.
Auxiliary equation: $r^2 + 12r + 32 = 0 \quad \rightarrow \quad (r+8)(r+4) = 0 \quad \rightarrow \quad r_1 = -8,\ r_2 = -4$, distinct real solutions.
Solution of form: $\boxed{y_c(x) = c_1e^{-8x} + c_2e^{-4x}.}$
2. We look for a particular solution of the equation $y'' + 12y' + 32y = \sin(e^{4x})$ of the form

$$y_p(x) = u_1(x)e^{-8x} + u_2(x)e^{-4x}.$$

3. We calculate $y_p'(x)$ to get

$$y_p'(x) = u_1'(x)e^{-8x} - 8u_1(x)e^{-8x} + u_2'(x)e^{-4x} - 4u_2(x)e^{-4x} = (u_1'(x)e^{-8x} + u_2'(x)e^{-4x}) - 8u_1(x)e^{-8x} - 4u_2(x)e^{-4x}.$$

4. We set $\boxed{u_1'(x)e^{-8x} + u_2'(x)e^{-4x} = 0}$.
This gives $y_p'(x) = -8u_1(x)e^{-8x} - 4u_2(x)e^{-4x}$.
5. Evaluate y_p'':

$$y_p''(x) = -8u_1'(x)e^{-8x} + 64u_1(x)e^{-8x} - 4u_2'(x)e^{-4x} + 16u_2(x)e^{-4x}.$$

Substitute $y_p'' = -8u_1'(x)e^{-8x} + 64u_1(x)e^{-8x} - 4u_2'(x)e^{-4x} + 16u_2(x)e^{-4x}$,
$y_p' = -8u_1(x)e^{-8x} - 4u_2(x)e^{-4x}$ and $y_p = u_1(x)e^{-8x} + u_2(x)e^{-4x}$ in the equation $y'' + 12y' + 32y = \sin(e^{4x})$ to get

$$-8u_1'(x)e^{-8x} + 64u_1(x)e^{-8x} - 4u_2'(x)e^{-4x} + 16u_2(x)e^{-4x} + 12[-8u_1(x)e^{-8x} - 4u_2(x)e^{-4x}] + 32[u_1(x)e^{-8x} + u_2(x)e^{-4x}]$$

$$= \sin(e^{4x}).$$

When we tidy up all terms except those with u_1' and u_2' should disappear (see book for details).
Tidying up gives:

$$-8u_1'(x)e^{-8x} - 4u_2'(x)e^{-4x} + (64 - 96 + 32)u_1(x)e^{-8x} + (16 - 48 + 32)u_2(x)e^{-4x} = \sin(e^{4x})$$

or

$$\boxed{-8u_1'(x)e^{-8x} - 4u_2'(x)e^{-4x} = \sin(e^{4x})}$$

6. Solve for u_1' and u_2' using both equations

$$u_1'(x)e^{-8x} + u_2'(x)e^{-4x} = 0, \qquad -8u_1'(x)e^{-8x} - 4u_2'(x)e^{-4x} = \sin(e^{4x}).$$

From the equation on the left, we get $u_1'(x) = -u_2'(x)e^{4x}$, substituting this into the equation on the right, we get

$$8u_2'(x)e^{-4x} - 4u_2'(x)e^{-4x} = \sin(e^{4x}) \quad \to \quad 4u_2'(x)e^{-4x} = \sin(e^{4x}) \quad \to \quad \boxed{u_2'(x) = \frac{e^{4x}\sin(e^{4x})}{4}}$$

$$u_1'(x) = -u_2'(x)e^{4x} \quad \to \quad u_1'(x) = -\frac{e^{4x}\sin(e^{4x})}{4}e^{4x} \quad \to \quad \boxed{u_1'(x) = -\frac{e^{8x}\sin(e^{4x})}{4}}$$

7. Integrate the resulting u_1' and u_2' to get functions u_1 and u_2.

$$u_2(x) = \int \frac{e^{4x}\sin(e^{4x})}{4}dx, \quad \text{let} \quad w = e^{4x}, \quad dw = 4e^{4x}dx, \quad e^{4x}dx = \frac{dw}{4}.$$

$$u_2(x) = \int \frac{e^{4x}\sin(e^{4x})}{4}dx = \frac{1}{16}\int \sin w \, dw = \frac{-\cos w}{16} = \frac{-\cos(e^{4x})}{16}.$$

$$\boxed{u_2(x) = \frac{-\cos(e^{4x})}{16}}$$

$$u_1(x) = \int -\frac{e^{8x}\sin(e^{4x})}{4}dx, \quad \text{let} \quad w = e^{4x}, \quad dw = 4e^{4x}dx, \quad e^{4x}dx = \frac{dw}{4}$$

$$u_1(x) = -\frac{1}{16}\int w\sin(w)\,dw, \quad \text{int. by parts} \quad u = w, \quad du = dw, \quad dv = \sin(w)dw \quad v = -cos(w)$$

$$u_1(x) = -\frac{1}{16}\int w\sin(w)\,dw = -\frac{1}{16}[-w\cos(w) + \int \cos(w)dw] = -\frac{1}{16}[-w\cos(w) + \sin(w)]$$

$$= -\frac{1}{16}[-e^{4x}\cos(e^{4x}) + \sin(e^{4x})]$$

$$\boxed{u_1(x) = -\frac{1}{16}[-e^{4x}\cos(e^{4x}) + \sin(e^{4x})]}$$

We have

$$y_p(x) = u_1(x)e^{-8x} + u_2(x)e^{-4x} = -\frac{1}{16}[-e^{4x}\cos(e^{4x}) + \sin(e^{4x})]e^{-8x} + [\frac{-\cos(e^{4x})}{16}]e^{-4x} = -\frac{1}{16}[e^{-8x}\sin(e^{4x})]$$

The general solution is given by

$$\boxed{y(x) = y_p(x) + y_c(x) = -\frac{1}{16}[e^{-8x}\sin(e^{4x})] + c_1e^{-8x} + c_2e^{-4x}.}$$