Lecture 25 : Integral Test

In this section, we see that we can sometimes decide whether a series converges or diverges by comparing it to an improper integral. The analysis in this section only applies to **series $\sum a_n$, with positive terms, that is $a_n > 0$.**

Integral Test Suppose $f(x)$ is a positive decreasing continuous function on the interval $[1, \infty)$ with $f(n) = a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is convergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is convergent.}$$

$$\text{If} \quad \int_1^{\infty} f(x)dx \quad \text{is divergent, then} \quad \sum_{n=1}^{\infty} a_n \quad \text{is divergent.}$$

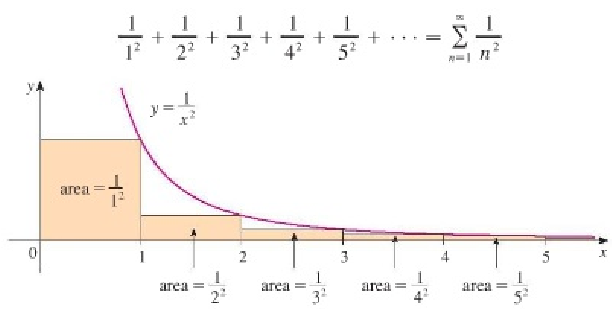
Note The result is still true if the condition that $f(x)$ is decreasing on the interval $[1, \infty)$ is relaxed to "the function $f(x)$ is decreasing on an interval $[M, \infty)$ for some number $M \geq 1$."

We can get some idea of the proof from the following examples:

We know from a previous lecture that

$$\int_1^{\infty} \frac{1}{x^p} dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leq 1.$$

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$.



We see that

$$s_n = 1 + \sum_{n=2}^{n} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2.$$

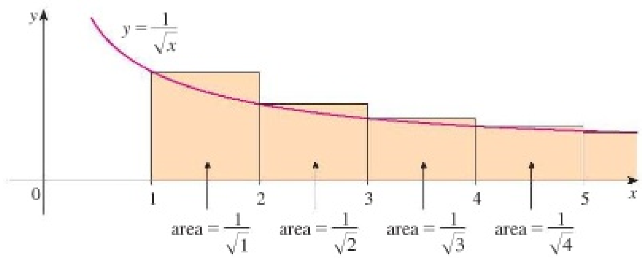
Since the sequence $\{s_n\}$ is increasing (because each $a_n > 0$) and bounded, we can conclude that the sequence of partial sums converges and hence the series

$$\sum_{i=1}^{\infty} \frac{1}{n^2} \quad \text{converges.}$$

NOTE We are not saying that $\sum_{i=1}^{\infty} \frac{1}{n^2} = \int_1^{\infty} \frac{1}{x^2} dx$ here.

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to the improper integral $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$



This time we draw the rectangles so that we get

$$s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}} dx$$

Thus we see that $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}} dx$. However, we know that $\int_1^n \frac{1}{\sqrt{x}} dx$ grows without bound and hence since $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ diverges, we can conclude that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

Example Use the integral test to determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{2}{3n+5}$$

Example Use the integral test to determine if the following series converges:

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

p-series

We can use the result quoted above from our section on improper integrals to prove the following result on the **p-series**, $\sum_{i=1}^{\infty} \frac{1}{n^p}$.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.}$$

Example Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} n^{-15}, \qquad \sum_{n=10}^{\infty} n^{-15}, \qquad \sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}},$$