In this section, we show how to use the integral test to decide whether a series of the form \( \sum_{n=a}^{\infty} \frac{1}{n^p} \) (where \( a \geq 1 \)) converges or diverges by comparing it to an improper integral. Series of this type are called p-series. We will in turn use our knowledge of p-series to determine whether other series converge or not by making comparisons (much like we did with improper integrals).

**Integral Test** Suppose \( f(x) \) is a positive decreasing continuous function on the interval \([1, \infty)\) with \( f(n) = a_n \). Then the series \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if \( \int_{1}^{\infty} f(x)dx \) converges, that is:

- If \( \int_{1}^{\infty} f(x)dx \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.
- If \( \int_{1}^{\infty} f(x)dx \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent.

**Note** The result is still true if the condition that \( f(x) \) is decreasing on the interval \([1, \infty)\) is relaxed to “the function \( f(x) \) is decreasing on an interval \([M, \infty)\) for some number \( M \geq 1\).”

We can get some idea of the proof from the following examples:

We know from our lecture on improper integrals that

\[
\int_{1}^{\infty} \frac{1}{x^p}dx \quad \text{converges if } p > 1 \quad \text{and diverges if } p \leq 1.
\]

**Example** In the picture below, we compare the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) to the improper integral \( \int_{1}^{\infty} \frac{1}{x^2}dx \).

We see that

\[
s_n = 1 + \sum_{n=2}^{n} \frac{1}{n^2} < 1 + \int_{1}^{\infty} \frac{1}{x^2}dx = 1 + 1 = 2.
\]

Since the sequence \( \{s_n\} \) is increasing (because each \( a_n > 0 \)) and bounded, we can conclude that the sequence of partial sums converges and hence the series

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]

converges.

**NOTE** We are not saying that \( \sum_{i=1}^{\infty} \frac{1}{n^2} = \int_{1}^{\infty} \frac{1}{x^2}dx \) here.
**Example**  In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to the improper integral $\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx$.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

This time we draw the rectangles so that we get

$$s_{n} > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_{1}^{n} \frac{1}{\sqrt{x}} \, dx$$

Thus we see that $\lim_{n \to \infty} s_{n} > \lim_{n \to \infty} \int_{1}^{n} \frac{1}{\sqrt{x}} \, dx$. However, we know that $\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx$ grows without bound and hence since $\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx$ diverges, we can conclude that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

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**p-series**

We can use the result quoted above from our section on improper integrals to prove the following result on the **p-series**, $\sum_{i=1}^{\infty} \frac{1}{n^p}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.$$  

**Example**  Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \quad \sum_{n=1}^{\infty} n^{-15}, \quad \sum_{n=10}^{\infty} n^{-15}, \quad \sum_{n=100}^{\infty} \frac{1}{\sqrt{n}}.$$  

2
Comparison Test

As we did with improper integral, we can compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

We will of course make use of our knowledge of $p$-series and geometric series.

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.
\]

\[
\sum_{n=1}^{\infty} ar^{n-1} \text{ converges if } |r| < 1, \text{ diverges if } |r| \geq 1.
\]

**Comparison Test**  Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, than $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is divergent.

**Proof** Let

\[
s_n = \sum_{i=1}^{n} a_i, \quad t_n = \sum_{i=1}^{n} b_i,
\]

Proof of (i):  Let us assume that $\sum b_n$ is convergent and that $a_n \leq b_n$ for all $n$. Both series have positive terms, hence both sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since we are assuming that $\sum_{n=1}^{\infty} b_n$ converges, we know that there exists a $t$ with $t = \sum_{n=1}^{\infty} b_n$. We have $s_n \leq t_n \leq t$ for all $n$. Hence since the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded above, it converges and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (ii):  Let us assume that $\sum b_n$ is divergent and that $a_n \geq b_n$ for all $n$. Since we are assuming that $\sum b_n$ diverges, we have the sequence of partial sums, $\{t_n\}$, is increasing and unbounded. Hence since we are assuming here that $a_n \geq b_n$ for each $n$, we have $s_n \geq t_n$ for each $n$. Thus the sequence of partial sums $\{s_n\}$ is unbounded and increasing and hence $\sum a_n$ diverges.
Example Use the comparison test to determine if the following series converge or diverge:

\[
\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}, \quad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2 + 1},
\]

\[
\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n!}
\]
Limit Comparison Test  Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. If
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]
where \( c \) is a finite number and \( c > 0 \), then either both series converge or both diverge.

Proof  Let \( m \) and \( M \) be numbers such that \( m < c < M \). Then, because \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \), there is an \( N \) for which \( m < \frac{a_n}{b_n} < M \) for all \( n > N \). This means that
\[
mb_n < a_n < Mb_n, \quad \text{when } n > N.
\]
Now we can use the comparison test from above to show that
If \( \sum a_n \) converges, then \( \sum mb_n \) also converges. Hence \( \frac{1}{m} \sum mb_n = \sum b_n \) converges.
On the other hand, if \( \sum b_n \) converges, then \( \sum Mb_n \) also converges and by comparison \( \sum a_n \) converges.

Example  Test the following series for convergence using the Limit Comparison test:
\[
\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}, \quad \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}, \quad \sum_{n=1}^{\infty} \frac{2n + 1}{\sqrt{n^3 + 1}}, \quad \sum_{n=1}^{\infty} \frac{e}{2^n - 1}, \quad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}, \quad \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^3, \quad \sum_{n=1}^{\infty} \sin \left( \frac{\pi}{n} \right).
\]