Lecture 25/26 : Integral Test for p-series and The Comparison test

In this section, we show how to use the integral test to decide whether a series of the form $\sum_{n=a}^{\infty} \frac{1}{n^p}$ (where $a \geq 1$) converges or diverges by comparing it to an improper integral. Serioes of this type are called p-series. We will in turn use our knowledge of p-series to determine whether other series converge or not by making comparisons (much like we did with improper integrals).

Integral Test Suppose $f(x)$ is a positive decreasing continuous function on the interval $[1, \infty)$ with $f(n) = a_n$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ converges, that is:

$$\text{If } \int_1^{\infty} f(x)dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\text{If } \int_1^{\infty} f(x)dx \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

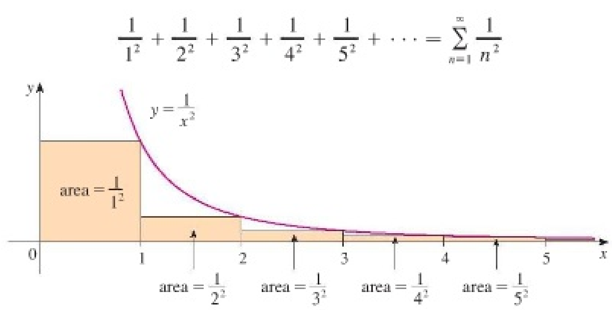
Note The result is still true if the condition that $f(x)$ is decreasing on the interval $[1, \infty)$ is relaxed to "the function $f(x)$ is decreasing on an interval $[M, \infty)$ for some number $M \geq 1$."

We can get some idea of the proof from the following examples:

We know from our lecture on improper integrals that

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ converges if } p > 1 \text{ and diverges if } p \leq 1.$$

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the improper integral $\int_1^{\infty} \frac{1}{x^2} dx$.



We see that

$$s_n = 1 + \sum_{n=2}^{n} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2.$$

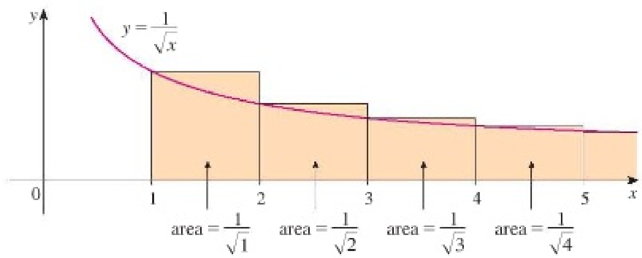
Since the sequence $\{s_n\}$ is increasing (because each $a_n > 0$) and bounded, we can conclude that the sequence of partial sums converges and hence the series

$$\sum_{i=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

NOTE We are not saying that $\sum_{i=1}^{\infty} \frac{1}{n^2} = \int_1^{\infty} \frac{1}{x^2} dx$ here.

Example In the picture below, we compare the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ to the improper integral $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$



This time we draw the rectangles so that we get

$$s_n > s_{n-1} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n-1}} > \int_1^n \frac{1}{\sqrt{x}} dx$$

Thus we see that $\lim_{n\to\infty} s_n > \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}} dx$. However, we know that $\int_1^n \frac{1}{\sqrt{x}} dx$ grows without bound and hence since $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ diverges, we can conclude that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges.

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p-series

We can use the result quoted above from our section on improper integrals to prove the following result on the **p-series**, $\sum_{i=1}^{\infty} \frac{1}{n^p}$.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{converges for} \quad p > 1, \quad \text{diverges for} \quad p \leq 1.}$$

Example Determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}, \qquad \sum_{n=1}^{\infty} n^{-15}, \qquad \sum_{n=10}^{\infty} n^{-15}, \qquad \sum_{n=100}^{\infty} \frac{1}{\sqrt[5]{n}},$$

Comparison Test

As we did with improper integral, we can compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

We will of course make use of our knowledge of p-series and geometric series.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.}$$

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} \text{ converges if } |r| < 1, \text{ diverges if } |r| \geq 1.}$$

Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, than $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Proof Let

$$s_n = \sum_{i=1}^{n} a_i, \quad t_n = \sum_{i=1}^{n} b_i,$$

Proof of (i): Let us assume that $\sum b_n$ is convergent and that $a_n \leq b_n$ for all n. Both series have positive terms, hence both sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since we are assuming that $\sum_{n=1}^{\infty} b_n$ converges, we know that there exists a t with $t = \sum_{n=1}^{\infty} b_n$. We have $s_n \leq t_n \leq t$ for all n. Hence since the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded above, it converges and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (ii): Let us assume that $\sum b_n$ is divergent and that $a_n \geq b_n$ for all n. Since we are assuming that $\sum b_n$ diverges, we have the sequence of partial sums, $\{t_n\}$, is increasing and unbounded. Hence since we are assuming here that $a_n \geq b_n$ for each n, we have $s_n \geq t_n$ for each n. Thus the sequence of partial sums $\{s_n\}$ is unbounded and increasing and hence $\sum a_n$ diverges.

Example Use the comparison test to determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

$$\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n!}$$

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof Let m and M be numbers such that $m < c < M$. Then, because $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there is an N for which $m < \frac{a_n}{b_n} < M$ for all $n > N$. This means that

$$mb_n < a_n < Mb_n, \quad \text{when} \;\; n > N.$$

Now we can use the comparison test from above to show that

If $\sum a_n$ converges, then $\sum mb_n$ also converges. Hence $\frac{1}{m}\sum mb_n = \sum b_n$ converges.

On the other hand, if $\sum b_n$ converges, then $\sum Mb_n$ also converges and by comparison $\sum a_n$ converges.

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^4+n^2+2n+1}, \qquad \sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+1}}, \qquad \sum_{n=1}^{\infty} \frac{e}{2^n-1},$$

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}, \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^3 3^{-n}, \qquad \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right).$$