Lecture 26: Comparison Test

In this section, as we did with improper integrals, we see how to compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

We will of course make use of our knowledge of $p$-series and geometric series.

\[ \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1. \]

\[ \sum_{n=1}^{\infty} ar^{n-1} \text{ converges if } |r| < 1, \text{ diverges if } |r| \geq 1. \]

**Comparison Test** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is divergent.

**Proof** Let

\[ s_n = \sum_{i=1}^{n} a_i, \quad t_n = \sum_{i=1}^{n} b_i, \]

Proof of (i): Let us assume that $\sum b_n$ is convergent and that $a_n \leq b_n$ for all $n$. Both series have positive terms, hence both sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since we are assuming that $\sum_{n=1}^{\infty} b_n$ converges, we know that there exists a $t = \sum_{n=1}^{\infty} b_n$. We have $s_n \leq t_n \leq t$ for all $n$. Hence since the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded above, it converges and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (ii): Let us assume that $\sum b_n$ is divergent and that $a_n \geq b_n$ for all $n$. Since we are assuming that $\sum b_n$ diverges, we have the sequence of partial sums, $\{t_n\}$, is increasing and unbounded. Hence since we are assuming here that $a_n \geq b_n$ for each $n$, we have $s_n \geq t_n$ for each $n$. Thus the sequence of partial sums $\{s_n\}$ is unbounded and increasing and hence $\sum a_n$ diverges.
Example Use the comparison test to determine if the following series converge or diverge:

\[
\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}, \quad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2 + 1},
\]

\[
\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n}, \quad \sum_{n=1}^{\infty} \frac{1}{n!}
\]
**Limit Comparison Test**  Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms. If

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]

where \( c \) is a finite number and \( c > 0 \), then either both series converge or both diverge.

**Proof**  Let \( m \) and \( M \) be numbers such that \( m < c < M \). Then, because \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \), there is an \( N \) for which \( m < \frac{a_n}{b_n} < M \) for all \( n > N \). This means that

\[
mb_n < a_n < Mb_n, \quad \text{when } n > N.
\]

Now we can use the comparison test from above to show that

If \( \sum a_n \) converges, then \( \sum mb_n \) also converges. Hence \( \frac{1}{m} \sum mb_n = \sum b_n \) converges.

On the other hand, if \( \sum b_n \) converges, then \( \sum Mb_n \) also converges and by comparison \( \sum a_n \) converges.

**Example**  Test the following series for convergence using the Limit Comparison test:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}, \quad \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 + n^2 + 2n + 1}, \quad \sum_{n=1}^{\infty} \frac{2n + 1}{\sqrt{n^3 + 1}}, \quad \sum_{n=1}^{\infty} \frac{e}{2^n - 1},
\]

\[
\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}, \quad \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^3, \quad \sum_{n=1}^{\infty} \sin \left(\frac{n^3}{n}\right).
\]