Lecture 26 :Comparison Test

In this section, as we did with improper integrals, we see how to compare a series (with Positive terms) to a well known series to determine if it converges or diverges.

We will of course make use of our knowledge of p-series and geometric series.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for } p > 1, \text{ diverges for } p \leq 1.}$$

$$\boxed{\sum_{n=1}^{\infty} ar^{n-1} \text{ converges if } |r| < 1, \text{ diverges if } |r| \geq 1.}$$

Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series **with positive terms.**

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, than $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Proof Let

$$s_n = \sum_{i=1}^{n} a_i, \quad t_n = \sum_{i=1}^{n} b_i,$$

Proof of (i): Let us assume that $\sum b_n$ is convergent and that $a_n \leq b_n$ for all n. Both series have positive terms, hence both sequences $\{s_n\}$ and $\{t_n\}$ are increasing. Since we are assuming that $\sum_{n=1}^{\infty} b_n$ converges, we know that there exists a t with $t = \sum_{n=1}^{\infty} b_n$. We have $s_n \leq t_n \leq t$ for all n. Hence since the sequence of partial sums for the series $\sum_{n=1}^{\infty} a_n$ is increasing and bounded above, it converges and hence the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof of (ii): Let us assume that $\sum b_n$ is divergent and that $a_n \geq b_n$ for all n. Since we are assuming that $\sum b_n$ diverges, we have the sequence of partial sums, $\{t_n\}$, is increasing and unbounded. Hence since we are assuming here that $a_n \geq b_n$ for each n, we have $s_n \geq t_n$ for each n. Thus the sequence of partial sums $\{s_n\}$ is unbounded and increasing and hence $\sum a_n$ diverges.

Example Use the comparison test to determine if the following series converge or diverge:

$$\sum_{n=1}^{\infty} \frac{2^{-1/n}}{n^3}, \qquad \sum_{n=1}^{\infty} \frac{2^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

$$\sum_{n=1}^{\infty} \frac{n^{-2}}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \sum_{n=1}^{\infty} \frac{1}{n!}$$

Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof Let m and M be numbers such that $m < c < M$. Then, because $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there is an N for which $m < \frac{a_n}{b_n} < M$ for all $n > N$. This means that

$$mb_n < a_n < Mb_n, \quad \text{when} \quad n > N.$$

Now we can use the comparison test from above to show that

If $\sum a_n$ converges, then $\sum mb_n$ also converges. Hence $\frac{1}{m}\sum mb_n = \sum b_n$ converges.

On the other hand, if $\sum b_n$ converges, then $\sum Mb_n$ also converges and by comparison $\sum a_n$ converges.

Example Test the following series for convergence using the Limit Comparison test:

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1} \qquad \sum_{n=1}^{\infty} \frac{n^2+2n+1}{n^4+n^2+2n+1}, \qquad \sum_{n=1}^{\infty} \frac{2n+1}{\sqrt{n^3+1}}, \qquad \sum_{n=1}^{\infty} \frac{e}{2^n-1},$$

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{n^2}, \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^3 3^{-n}, \qquad \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right).$$