Lecture 3 : The Natural Exponential Function: \( f(x) = \exp(x) = e^x \)

Last day, we saw that the function \( f(x) = \ln x \) is one-to-one, with domain \((0, \infty)\) and range \((-\infty, \infty)\). We can conclude that \( f(x) \) has an inverse function \( f^{-1}(x) = \exp(x) \) which we call the natural exponential function. The definition of inverse functions gives us the following:

\[
y = f^{-1}(x) \quad \text{if and only if} \quad x = f(y)
\]

\[
y = \exp(x) \quad \text{if and only if} \quad x = \ln(y)
\]

The cancellation laws give us:

\[
f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x
\]

\[
\exp(\ln x) = x \quad \text{and} \quad \ln(\exp(x)) = x
\]

We can draw the graph of \( y = \exp(x) \) by reflecting the graph of \( y = \ln(x) \) in the line \( y = x \).

We have that the graph \( y = \exp(x) \) is one-to-one and continuous with domain \((-\infty, \infty)\) and range \((0, \infty)\). Note that \( \exp(x) > 0 \) for all values of \( x \). We see that

\[
\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0
\]
\[
\exp(1) = e \quad \text{since} \quad \ln e = 1,
\]
\[
\exp(2) = e^2 \quad \text{since} \quad \ln(e^2) = 2,
\]
\[
\exp(-7) = e^{-7} \quad \text{since} \quad \ln(e^{-7}) = -7.
\]

In fact for any rational number \( r \), we have

\[
\exp(r) = e^r \quad \text{since} \quad \ln(e^r) = r \ln e = r,
\]
by the laws of Logarithms.

When \( x \) is rational or irrational, we define \( e^x \) to be \( \exp(x) \).

\[
e^x = \exp(x)
\]

**Note:** This agrees with definitions of \( e^x \) given elsewhere, since the definition is the same when \( x \) is a rational number and the exponential function is continuous.

Restating the above properties given above in light of this new interpretation of the exponential function, we get:

\[
\begin{align*}
e^x &= y \text{ if and only if } \ln y = x \\
e^{\ln x} &= x \quad \text{and} \quad \ln e^x &= x
\end{align*}
\]

**Solving Equations**

We can use these formulas to solve equations.

**Example** Solve for \( x \) if \( \ln(x + 1) = 5 \)

**Example** Solve for \( x \) if \( e^{x-4} = 10 \)

**Limits**

From the graph we see that

\[
\lim_{x \to -\infty} e^x = 0, \quad \lim_{x \to \infty} e^x = \infty.
\]

**Example** Find the limit \( \lim_{x \to \infty} \frac{e^x}{10e^x-1} \).
Rules of Exponents

The following rules of exponents follow from the rules of logarithms:

\[ e^{x+y} = e^x e^y, \quad e^{x-y} = \frac{e^x}{e^y}, \quad (e^x)^y = e^{xy}. \]

**Proof**  We have \( \ln(e^{x+y}) = x + y = \ln(e^x) + \ln(e^y) = \ln(e^xe^y) \). Therefore \( e^{x+y} = e^x e^y \). The other rules can be proven similarly.

**Example** Simplify \( \frac{e^x e^{2x+1}}{(e^x)^2} \).

Derivatives

\[
\begin{align*}
\frac{d}{dx} e^x &= e^x \\
\frac{d}{dx} e^{g(x)} &= g'(x)e^{g(x)}
\end{align*}
\]

**Proof**  We use logarithmic differentiation. If \( y = e^x \), we have \( \ln y = x \) and differentiating, we get \( \frac{1}{y} \frac{dy}{dx} = 1 \) or \( \frac{dy}{dx} = y = e^x \). The derivative on the right follows from the chain rule.

**Example** Find \( \frac{d}{dx} e^{\sin^2 x} \) and \( \frac{d}{dx} \sin^2(e^{x^2}) \).

Integrals

\[
\begin{align*}
\int e^x \, dx &= e^x + C \\
\int g'(x)e^{g(x)} \, dx &= e^{g(x)} + C
\end{align*}
\]

**Example** Find \( \int x e^{x^2+1} \, dx \).
Old Exam Questions

Old Exam Question The function \( f(x) = x^3 + 3x + e^{2x} \) is one-to-one. Compute \( f^{-1}'(1) \).

Old Exam Question Compute the limit
\[
\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.
\]

Old Exam Question Compute the Integral
\[
\int_0^{\ln 2} \frac{e^x}{1 + e^x} \, dx.
\]
Extra Examples (please attempt these before looking at the solutions)

Example Find the domain of the function \( g(x) = \sqrt{50 - e^x} \).

Example Solve for \( x \) if \( \ln(\ln(x^2)) = 10 \)

Example Let \( f(x) = e^{4x+3} \), Show that \( f \) is a one-to-one function and find the formula for \( f^{-1}(x) \).

Example Evaluate the integral
\[
\int_{3e^2}^{3e^4} \frac{1}{x\left(\ln \left( \frac{x}{3} \right) \right)^3} \, dx.
\]

Example Find the limit \( \lim_{x \to -\infty} \frac{e^x}{10e^x - 1} \) and \( \lim_{x \to 0} \frac{e^x}{10e^x - 1} \).

Example Find \( \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x) e^{\sin x} \, dx \).

Example Find the first and second derivatives of \( h(x) = \frac{e^x}{10e^x - 1} \). Sketch the graph of \( h(x) \) with horizontal, and vertical asymptotes, showing where the function is increasing and decreasing and showing intervals of concavity and convexity.
**Example** Find the domain of the function \( g(x) = \sqrt{50 - e^x} \).

The domain of \( g \) is \( \{ x | 50 - e^x \geq 0 \} \).

\[
50 - e^x \geq 0 \quad \text{if and only if} \quad 50 \geq e^x \\
\text{if and only if} \quad \ln 50 \geq \ln(e^x) = x \quad \text{or} \quad x \leq \ln 50
\]

since \( \ln(x) \) is an increasing function.

**Example** Solve for \( x \) if \( \ln(\ln(x^2)) = 10 \)

We apply the exponential function to both sides to get

\[
e^{\ln(\ln(x^2))} = e^{10} \quad \text{or} \quad \ln(x^2) = e^{10}.
\]

Applying the exponential function to both sides again, we get

\[
e^{\ln(x^2)} = e^{e^{10}} \quad \text{or} \quad x^2 = e^{e^{10}}.
\]

Taking the square root of both sides, we get

\[
x = \sqrt{e^{e^{10}}}.\]

**Example** Let \( f(x) = e^{4x+3} \), Show that \( f \) is a one-to-one function and find the formula for \( f^{-1}(x) \).

We have the domain of \( f \) is all real numbers. To find a formula for \( f^{-1} \), we use the method given in lecture 1.

\[
y = e^{4x+3} \quad \text{is the same as} \quad x = f^{-1}(y).
\]

we solve for \( x \) in the equation on the left, first we apply the logarithm function to both sides

\[
\ln(y) = \ln(e^{4x+3}) = 4x + 3 \quad \rightarrow \quad 4x = \ln(y) - 3 \quad \rightarrow \quad x = \frac{\ln(y) - 3}{4} = f^{-1}(y).
\]

Now we switch the \( x \) and \( y \) to get

\[
y = \frac{\ln(x) - 3}{4} = f^{-1}(x).
\]

**Example** Evaluate the integral

\[
\int_{3e^2}^{3e^4} \frac{1}{x \left( \ln \frac{x}{3} \right)^3} \, dx.
\]

We try the substitution \( u = \ln \frac{x}{3} \).

\[
du = \frac{3}{x} \cdot \frac{1}{3} \, dx = \frac{1}{x} \, dx, \quad u(3e^2) = 2, \quad u(3e^4) = 4.
\]

\[
\int_{3e^2}^{3e^4} \frac{1}{x \left( \ln \frac{x}{3} \right)^3} \, dx = \int_{2}^{4} \frac{1}{u^3} \, du = \left. \frac{u^{-2}}{-2} \right|_{2}^{4} = \frac{1}{-2u^2} \bigg|_{2}^{4}
\]
We use substitution. Let \( y = \frac{e^x}{10e^x - 1} \).

\[
\frac{1}{(-2)(16)} - \frac{1}{(-2)(4)} = \frac{1}{8} - \frac{1}{32} = \frac{3}{32}
\]

**Example** Find the limit \( \lim_{x \to -\infty} \frac{e^x}{10e^x - 1} \) and \( \lim_{x \to 0} \frac{e^x}{10e^x - 1} \).

\[
\lim_{x \to -\infty} \frac{e^x}{10e^x - 1} = \frac{\lim_{x \to -\infty} e^x}{\lim_{x \to -\infty} (10e^x - 1)} = 0 - 1 = 0.
\]

\[
\lim_{x \to 0} \frac{e^x}{10e^x - 1} = \frac{\lim_{x \to 0} e^x}{\lim_{x \to 0} (10e^x - 1)} = \frac{1}{10 - 1} = \frac{1}{9}.
\]

**Example** Find \( \int_0^2 (\cos x)e^{\sin x} \, dx \).

We use substitution. Let \( u = \sin x \), then \( du = \cos x \, dx \), \( u(0) = 0 \) and \( u(\pi/2) = 1 \).

\[
\int_0^2 (\cos x)e^{\sin x} \, dx = \int_0^1 e^u \, du = e^u \bigg|_0^1 = e^1 - e^0 = e - 1.
\]

**Example** Find the first and second derivatives of \( h(x) = \frac{e^x}{10e^x - 1} \). Sketch the graph of \( h(x) \) with horizontal, and vertical asymptotes, showing where the function is increasing and decreasing and showing intervals of concavity and convexity.

**y-int:** \( h(0) = \frac{1}{9} \)

**x-int:** \( h(x) = 0 \) if and only if \( e^x = 0 \), this is impossible, so there is no \( x \) intercept.

**H.A.:** In class, we saw \( \lim_{x \to -\infty} \frac{e^x}{10e^x - 1} = \frac{1}{10} \) and above, we saw \( \lim_{x \to -\infty} \frac{e^x}{10e^x - 1} = 0 \).

So the H.A.’s are \( y = 0 \) and \( y = \frac{1}{10} \).

**V.A.:** The graph has a vertical asymptote at \( x \) if \( 10e^x = 1 \), that is if \( e^x = \frac{1}{10} \) or \( x = \ln(\frac{1}{10}) \).

**Inc/Dec (h'(x))** To determine where the graph is increasing or decreasing, we calculate the derivative using the quotient rule

\[
h'(x) = \frac{(10e^x - 1)e^x - e^x(10e^x)}{(10e^x - 1)^2} = \frac{e^x(10e^x - 1 - 10e^x)}{(10e^x - 1)^2} = \frac{-e^x}{(10e^x - 1)^2}.
\]

Since \( h'(x) \) is always negative, the graph of \( y = h(x) \) is always decreasing.

**Concave/Convex** To determine intervals of concavity and convexity, we calculate the second derivative.

\[
h''(x) = \frac{d}{dx} h'(x) = \frac{d}{dx} \left( \frac{-e^x}{(10e^x - 1)^2} \right) = -\frac{d}{dx} \left( \frac{e^x}{(10e^x - 1)^2} \right).
\]

I’m going to use logarithmic differentiation here

\[
y = \frac{e^x}{(10e^x - 1)^2} \quad \rightarrow \quad \ln(y) = \ln(e^x) - 2 \ln(10e^x - 1) = x - 2 \ln(10e^x - 1)
\]

differentiating both sides, we get

\[
\frac{1}{y} \frac{dy}{dx} = 1 - 2 \cdot \frac{1}{10e^x - 1} \cdot 10e^x = 1 - \frac{20e^x}{10e^x - 1}.
\]

Multiplying across by \( y = \frac{e^x}{(10e^x - 1)^2} \), we get

\[
\frac{dy}{dx} = \frac{e^x}{(10e^x - 1)^2} - \frac{e^x}{(10e^x - 1)^2} \cdot \frac{20e^x}{10e^x - 1}.
\]

\[
\frac{dy}{dx} = \frac{e^x}{(10e^x - 1)^2} - \frac{20e^x}{10e^x - 1}.
\]
\[
\frac{e^x (10e^x - 1) - e^x (20e^x)}{(10e^x - 1)^3} = \frac{e^x (-1 - 10e^x)}{(10e^x - 1)^3} = \frac{-e^x (1 + 10e^x)}{(10e^x - 1)^3}
\]

\[
h''(x) = -\frac{dy}{dx} = \frac{e^x (1 + 10e^x)}{(10e^x - 1)^3}
\]

We see that the numerator is always positive here. From our calculations above, we have \(10e^x - 1 < 0\) if \(x < \ln(1/10)\) and \(10e^x - 1 > 0\) if \(x > \ln(1/10)\).

Therefore \(h''(x) < 0\) if \(x < \ln(1/10)\) and \(h''(x) > 0\) if \(x > \ln(1/10)\) and

The graph of \(y = h(x)\) is concave down if \(x < \ln(1/10)\) and concave up if \(x > \ln(1/10)\).

Putting all of this together, you should get a graph that looks like:

Check it and other functions out in Mthematica

\[
\ln(2) = N[\text{Log}[1/10]]
\]
\[
\text{Out}[2] = -2.30259
\]

\[
\ln(3) = \text{Plot}[\text{Exp}[x]/(10 \text{Exp}[x] - 1),
\{x, -10, 10\}]
\]
\[
\text{Out}[3] = \text{Plot}[\text{Exp}[x]/(10 \text{Exp}[x] - 1),
\{x, -10, 10\}]
\]


**Answers to Old Exam Questions**

**Old Exam Question** The function \( f(x) = x^3 + 3x + e^{2x} \) is one-to-one. Compute \( f^{-1}'(1) \).

We use the formula

\[
(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}
\]

\( b = f^{-1}(1) \) same as \( f(b) = 1 \rightarrow b^3 + 3b + e^{2b} = 1. \)

Solving for \( b \) is very difficult, but we can work by trail and error. If we try \( b = 0 \), we see that it works, since \( e^0 = 1 \). Therefore \( f^{-1}(1) = 0. \)

We also need to calculate \( f'(x) \), we get \( f'(x) = 3x^2 + 3 + 2e^{2x}. \)

\[
(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{3 + 2} = \frac{1}{5}.
\]

**Old Exam Question** Compute the limit

\[
\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}}.
\]

We divide both numerator and denominator by the highest power of \( e^x \) in the denominator which is \( e^{2x} \) in this case.

\[
\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^{2x} - e^{-2x}} = \lim_{x \to \infty} \frac{(e^x - e^{-x})/e^{2x}}{(e^{2x} - e^{-2x})/e^{2x}} = \lim_{x \to \infty} \frac{e^{-x} - e^{-3x}}{1 - e^{-4x}} = 0.
\]

**Old Exam Question** Compute the Integral

\[
\int_{0}^{\ln 2} \frac{e^x}{1 + e^x} dx.
\]

We make the substitution \( u = 1 + e^x \). We have

\[
du = e^x dx, \quad u(0) = 2, \quad u(\ln 2) = 1 + e^{\ln 2} = 3.
\]

We get

\[
\int_{0}^{\ln 2} \frac{e^x}{1 + e^x} dx = \int_{2}^{3} \frac{1}{u} du = \ln |u| \bigg|_{2}^{3} = \ln 3 - \ln 2 = \ln(3/2).
\]