

## Lecture 31: Power series representations of Functions

From our knowledge of Geometric Series, we know that

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1.$$

( Recall that we had

$$a + ar + ar^2 + ar^3 + \dots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } -1 < r < 1$$

and this series diverges for  $|r| \geq 1$ .

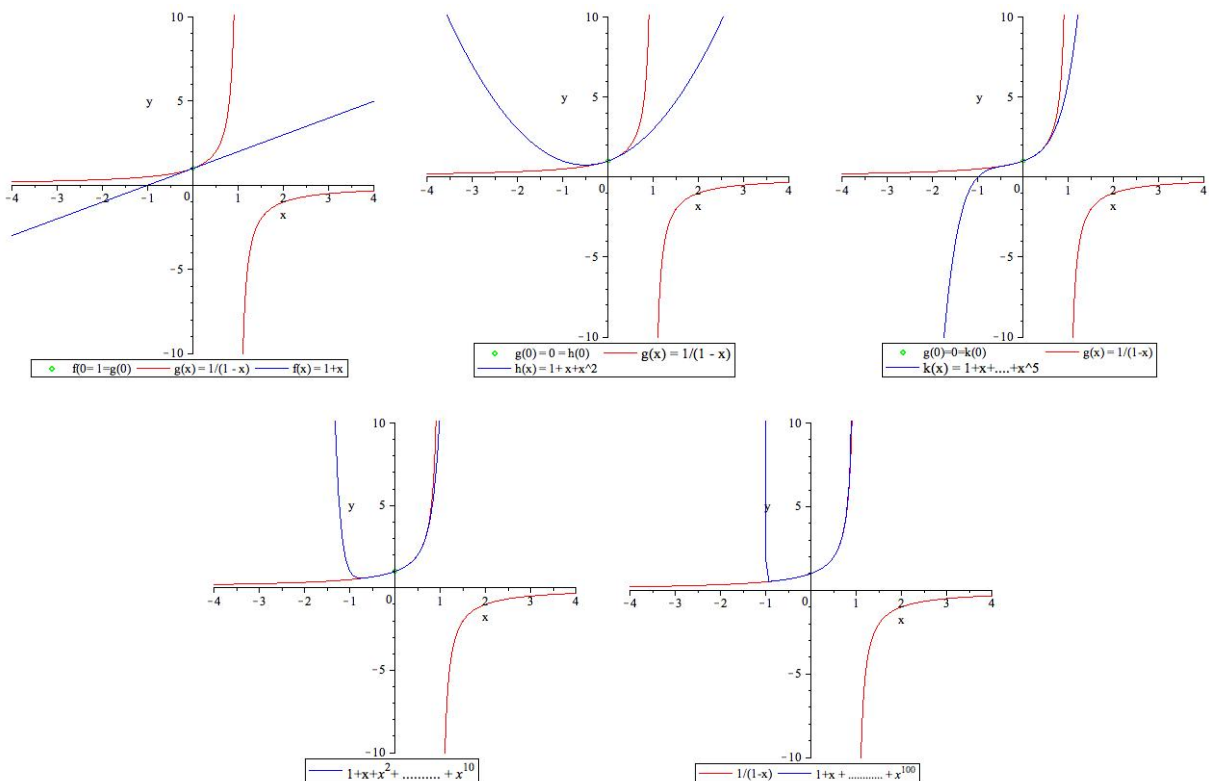
Above we have  $a = 1$  and  $x = r$ . )

This gives us a **power series representation for the function  $g(x)$  on the interval  $(-1, 1)$** .

**Note** that the function  $g(x)$  here has a larger domain than the power series.

The  $n$ th partial sum of the above power series is given by  $P_n(x) = 1 + x + x^2 + x^3 + \dots + x^n$ .

Hence, as  $n \rightarrow \infty$ , the graphs of the polynomials,  $P_n(x) = 1 + x + x^2 + x^3 + \dots + x^n$  get closer to the graph of  $f(x)$  on the interval  $(-1, 1)$ .



Having a power series representation of a function on an interval is useful for the purposes of integration, differentiation and solving differential equations.

### Method of Substitution

First, we examine how to use the power series representation of the function  $g(x) = 1/(1-x)$  on the interval  $(-1, 1)$  to derive a power series representation of other functions on an interval.

**Example (Substitution)** Find a power series representation of the functions given below and find the interval of convergence of the series.

$$f(x) = \frac{1}{1 + x^7},$$

**Example (Substitution)** Find a power series representation of the functions given below and find the interval of convergence of the series.

$$f(x) = \frac{2x}{1 + x},$$

**Example (Substitution)** Find a power series representation of the functions given below and find the interval of convergence of the series

$$h(x) = \frac{1}{4 - x}$$

## Differentiation and Integration of Power Series

We can differentiate and integrate power series term by term, just as we do with polynomials.

**Theorem** If the series  $\sum c_n(x-a)^n$  has radius of convergence  $R > 0$ , then the function  $f$  defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval  $(a-R, a+R)$  and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Also

$$\int f(x)dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of both of these power series is also  $R$ . (The interval of convergence may not remain the same when a series is differentiated or integrated; in particular convergence or divergence may change at the end points).

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**Example** Find a power series representation of the function

$$\frac{1}{(x+1)^2}.$$

**Example (Integration)** Find a power series representation of the function

$$\ln(1+x).$$

**Extra (Summing Series)** Use the fact that a power series (with  $x$  values in the real numbers) is continuous on its domain to show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln(2).$$

We have

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \text{for } -1 < x < 1.$$

When  $x = -1$ , the series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$  which diverges, by comparison with the harmonic series.

When  $x = 1$ , the series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  which converges, by the alternating series test.

Therefore the interval of convergence of the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$  is  $(-1, 1]$  and since this power series is continuous on its interval of convergence, we have

$$\lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

. Using the fact derived above that  $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \ln(1+x)$  for  $x < 1$ , we have

$$\lim_{x \rightarrow 1^-} \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

or

$$\ln(2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}.$$

**Example (Approximation)** Use power series to approximate the following integral up to 4 decimal places:

$$\int_0^{0.1} \frac{1}{1+x^7} dx$$

**Extra Example (Integration)** Show that

$$\tan^{-1}(x) = \int \frac{1}{1+x^2} dx$$

and use your answer calculate  $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}$ .

Using the power series representation of  $\frac{1}{1-x}$  on the interval  $(-1, 1)$ , we get a power series representation of  $\frac{1}{1+x^2}$ :

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots \quad \text{on the interval } (-1, 1).$$

Now we can integrate term by term to get a power series representation of  $\tan^{-1}(x)$  on the interval  $(-1, 1)$ ,

$$\tan^{-1}(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \dots \quad \text{on the interval } (-1, 1).$$

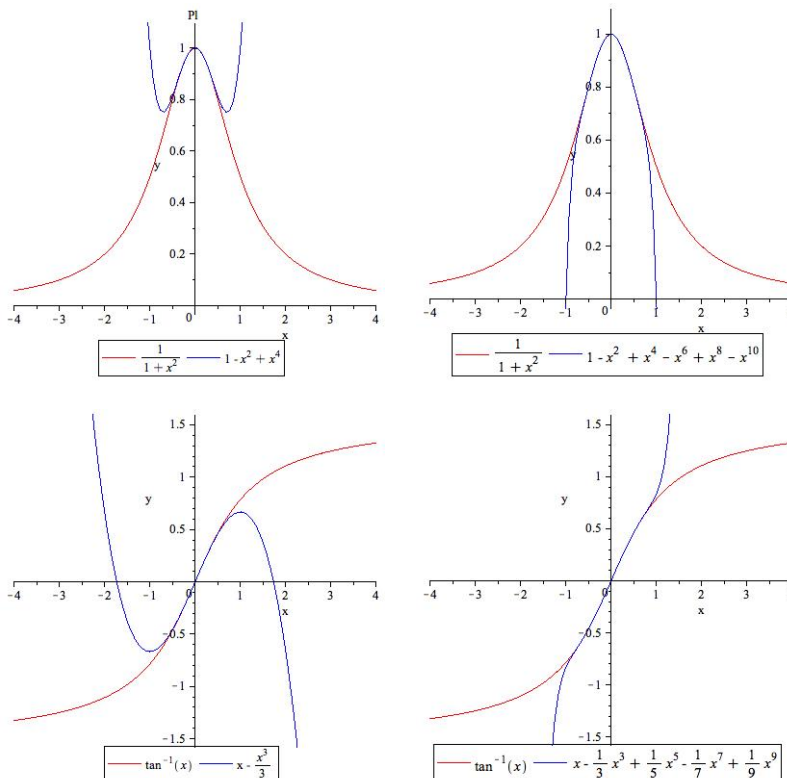
Since  $\tan^{-1}(0) = 0$ , we have  $C = 0$  and

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} - \dots \quad \text{on the interval } (-1, 1).$$

Since  $\frac{1}{\sqrt{3}} < 1$  and with  $x = \frac{1}{\sqrt{3}}$ , we get

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{3}^{(2n+1)}(2n+1)}.$$

The pictures shown below are of the 4th and 10th partial sums of the above two series, alongside the graphs of the corresponding functions.



**Extra Example:( Substitution)** Find a power series representation of the function

$$\tan^{-1}\left(\frac{x}{2}\right)$$

We saw above that

$$\tan^{-1}(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1} = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} - \dots \quad \text{on the interval } (-1, 1).$$

Letting  $y = \frac{x}{2}$ , we get

$$\tan^{-1}\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+1}}{2n+1} = (x/2) - \frac{(x/2)^3}{3} + \frac{(x/2)^5}{5} - \frac{(x/2)^7}{7} - \dots \quad \text{for } -1 < \frac{x}{2} < 1.$$

giving us that

$$\tan^{-1}\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1) \cdot 2^{2n+1}} = \left[ \frac{x}{2} - \frac{x^3}{3 \cdot 2^3} + \frac{x^5}{5 \cdot 2^5} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1) \cdot 2^{2n+1}} + \dots \right]$$

for  $-2 < x < 2$ .



**Extra Example; Two methods, same result**

Find a power series representation of the function

$$\ln(x^2 + 4)$$

First Way: Substitution and Integration We have

$$\ln(x^2 + 4) = \int \frac{2x}{x^2 + 4} dx$$

Check that

$$\frac{1}{4 + x^2} = \left[ \frac{1}{4} - \frac{x^2}{4^2} + \frac{x^4}{4^3} - \frac{x^6}{4^4} + \dots + \frac{(-1)^n x^{2n}}{4^{n+1}} + \dots \right] = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}} \right] \quad \text{for } -2 < x < 2$$

Hence

$$\frac{2x}{4 + x^2} = \left[ \frac{2x}{4} - \frac{2x \cdot x^2}{4^2} + \frac{2x \cdot x^4}{4^3} - \frac{2x \cdot x^6}{4^4} + \dots + \frac{(-1)^n 2x \cdot x^{2n}}{4^{n+1}} + \dots \right] = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2x \cdot x^{2n}}{4^{n+1}} \right] \quad \text{for } -2 < x < 2$$

$$\left[ \frac{2x}{4} - \frac{2x^3}{4^2} + \frac{2x^5}{4^3} - \frac{2x^7}{4^4} + \dots + \frac{(-1)^n 2x^{2n+1}}{4^{n+1}} + \dots \right] = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+1}}{4^{n+1}} \right] \quad \text{for } -2 < x < 2$$

Integrating both sides, we get

$$\begin{aligned} \ln(x^2+4) &= \int \frac{2x}{x^2 + 4} dx = C + \left[ \int \frac{2x}{4} dx - \int \frac{2x^3}{4^2} dx + \int \frac{2x^5}{4^3} dx - \int \frac{2x^7}{4^4} dx + \dots + \int \frac{(-1)^n 2x^{2n+1}}{4^{n+1}} dx + \dots \right] \\ &= C + \left[ \sum_{n=0}^{\infty} \int \frac{(-1)^n 2x^{2n+1}}{4^{n+1}} dx \right] \quad \text{for } -2 < x < 2. \end{aligned}$$

Substituting  $x = 0$  into the equation, we get  $\ln 4 = C$ . Thus we get

$$\begin{aligned} \ln(x^2 + 4) &= \ln 4 + \left[ \frac{2x^2}{2 \cdot 4} - \frac{2x^4}{4 \cdot 4^2} + \frac{2x^6}{6 \cdot 4^3} - \frac{2x^8}{8 \cdot 4^4} + \dots + \frac{(-1)^n 2x^{2n+2}}{(2n+2) \cdot 4^{n+1}} + \dots \right] \\ &= \ln 4 + \left[ \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2}}{(2n+2) \cdot 4^{n+1}} \right] \quad \text{for } -2 < x < 2 \\ &= \ln 4 + \left[ \frac{x^2}{4} - \frac{x^4}{2 \cdot 4^2} + \frac{x^6}{3 \cdot 4^3} - \frac{x^8}{4 \cdot 4^4} + \dots + \frac{(-1)^n x^{2n+2}}{(n+1) \cdot 4^{n+1}} + \dots \right] = \boxed{\ln 4 + \left[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n+1) \cdot 4^{n+1}} \right]} \quad \text{for } -2 < x < 2 \end{aligned}$$

Second Way (Substitution):  $\ln(4 + x^2) = \ln(4(1 + \frac{x^2}{4})) = \ln(4) + \ln(1 + \frac{x^2}{4})$ . We can now use our result from before

$$\ln(1 + y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{n+1}}{n+1}.$$

Let  $y = \frac{x^2}{4}$  to get

$$\ln\left(1 + \frac{x^2}{4}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2/4)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^{n+1}(n+1)}$$

Hence

$$\boxed{\ln(4 + x^2) = \ln(4) + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{4^{n+1}(n+1)}}.$$

**Extra Example: (Substitution)** Find a power series representation of the functions given below and find the interval of convergence of the series.

$$g(x) = \frac{2x^2}{3-x},$$

We have

$$\frac{2x^2}{3-x} = \frac{2x^2}{3} \left[ \frac{1}{1-(x/3)} \right].$$

Now recall from above that

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots + y^n + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for } -1 < y < 1$$

Therefore, substituting  $x/3$  for  $y$ , we get

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots + \left(\frac{x}{3}\right)^n + \cdots = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \quad \text{for } -1 < \left(\frac{x}{3}\right) < 1$$

We have  $-1 < \left(\frac{x}{3}\right) < 1$  if  $-3 < x < 3$  (multiplying the inequality by 3). Therefore

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{3^n} \quad \text{for } -3 < x < 3.$$

Now we want a power series representation for

$$g(x) = \frac{2x^2}{3-x} = \frac{2x^2}{3} \left[ \frac{1}{1-(x/3)} \right]$$

using the power series derived above for  $\frac{1}{1-(x/3)}$ , we get

$$\frac{2x^2}{3-x} = \frac{2x^2}{3} \left[ 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots \right] = \frac{2x^2}{3} \sum_{n=0}^{\infty} \frac{x^n}{3^n} \quad \text{for } -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[ \frac{2x^2}{3} + \frac{2x^2}{3} \left(\frac{x}{3}\right) + \frac{2x^2}{3} \left(\frac{x^2}{3^2}\right) + \frac{2x^2}{3} \left(\frac{x^3}{3^3}\right) + \cdots + \frac{2x^2}{3} \left(\frac{x^n}{3^n}\right) + \cdots \right] = \sum_{n=0}^{\infty} \frac{2x^2}{3} \left(\frac{x^n}{3^n}\right) \quad \text{for } -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[ \frac{2x^2}{3} + \frac{2x^3}{3^2} + \frac{2x^4}{3^3} + \frac{2x^5}{3^4} + \cdots + \frac{2x^{n+2}}{3^{n+1}} + \cdots \right] = \sum_{n=0}^{\infty} \frac{2x^{n+2}}{3^{n+1}} \quad \text{for } -3 < x < 3.$$