## Lecture 34: Curves Defined by Parametric Equations

When the path of a particle moving in the plane is not the graph of a function, we cannot describe it using a formula that express $y$ directly in terms of $x$, or $x$ directly in terms of $y$. Instead, we need to use a third variable $t$, called a **parameter** and write:

$$x = f(t) \qquad y = g(t)$$

The set of points $(x, y) = (f(t), g(t))$ described by these equations when $t$ varies in an interval $I$ form a curve, called a **parametric curve**, and $x = f(t), y = g(t)$ are called the **parametric equations** of the curve. Often, $t$ represents time and therefore we can think of $(x, y) = (f(t), g(t))$ as the position of a particle at time $t$.

If $I$ is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** and the point $(f(b), g(b))$ is the **terminal point**.

**Example 1** Draw and identify the parametric curve given by the parametric equations:

$$x = \cos t \qquad y = \sin t \qquad 0 \leq t \leq 2\pi$$

| $t$ | $x$ | $y$ |
|---|---|---|
| $0$ | | |
| $\frac{\pi}{4}$ | | |
| $\frac{\pi}{2}$ | | |
| $\frac{3\pi}{4}$ | | |
| $\pi$ | | |
| $\frac{3\pi}{2}$ | | |
| $2\pi$ | | |

As $t$ increases, we travel along the curve in a particular direction giving the curve an **orientation** which is often indicated by arrows. The curve however may pass through a section of the curve repeatedly and may do so in different directions for different values of $t$.

**Example 2** Describe the parametric curve represented by the parametric equations:

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq 2\pi$$

| $t$ | $x$ | $y$ |
|---|---|---|
| $0$ | | |
| $\frac{\pi}{4}$ | | |
| $\frac{\pi}{2}$ | | |
| $\frac{3\pi}{4}$ | | |
| $\pi$ | | |
| $\frac{3\pi}{2}$ | | |
| $2\pi$ | | |

**Note** The curve in examples 1 and 2 are the same but the parametric curve are not. Because in one case the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from $(1, 0)$. In example 2 instead, the point $(x, y) = (\sin 2t, \cos 2t)$ moves twice around the circle in the clockwise direction starting from $(0, 1)$.

**Example 3** Sketch the graph of the curve described by the following set of parametric equations.

$$x = t^3 - t, \quad y = t^2, \quad 0 \leq t < \infty$$

| $t$ | $x$ | $y$ |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1 | | |
| 1.5 | | |
| 2 | | |

**Converting from parametric equations to equations in Cartesian co-ordinates.**

There is no exact method for converting parametric equations for a curve to an equation in $x$ and $y$ only. If we can solve for $t$ in terms of either $x$ or $y$, we can substitute this for the value of $t$ in one of the equations to get an equation in $x$ and $y$ only.

**Example 4** Covert the following parametric equation to an equation relating $x$ and $y$:

$$x = 2t + 1, \quad y = t - 2.$$

and describe the curve traced as $t$ runs from $-\infty$ to $\infty$.

Sometimes, we can see a relationship between the $x$ and $y$ co-ordinates and thus eliminate the $t$.

**Example 5** Convert the following parametric equation to an equation relating $x$ and $y$:

$$x = 2\cos t \qquad y = 3\sin t$$

and describe the curve traced when $0 \leq t \leq 4\pi$.

## Converting from equations in Cartesian co-ordinates to parametric equations

**Easy cases** If a curve is defined by the equation $y = f(x)$, the equations $x = t$ and $y = f(t)$ give parametric equations describing the curve.

If a curve is described by the equation $x = g(y)$, the equations $x = t$ and $x = g(t)$ give parametric equations describing the curve.

**Example 6** Give parametric equations describing the graph of the parabola $y = x^2$.

**Example 7** Find parametric equations on $0 \leq t \leq 2\pi$ for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$ twice counterclockwise.

## Extra examples, see computer graphs or plot some points

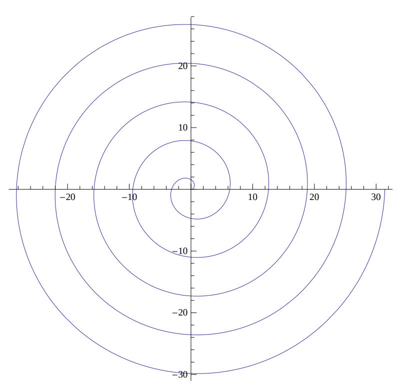
### Spiral

The following equations describe a spiral with an anticlockwise direction:

$$x = t \cos t, \quad y = t \sin t.$$

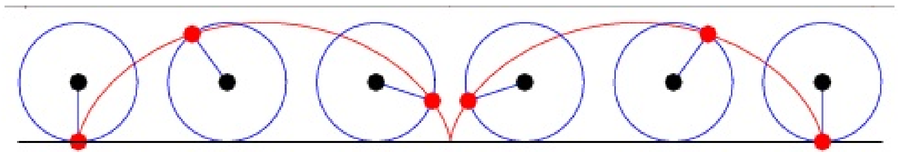
The graph shown below can be drawn in Mathematica with the following command:

```
In[13]:= ParametricPlot[{t*Cos[t], t*Sin[t]},
          {t, 0, 10*Pi}]
```



## The cycloid

The curve traced out by a point $P$ on the circumference of a circle of radius $r$ as the circle rolls along a straight line is called a cycloid.

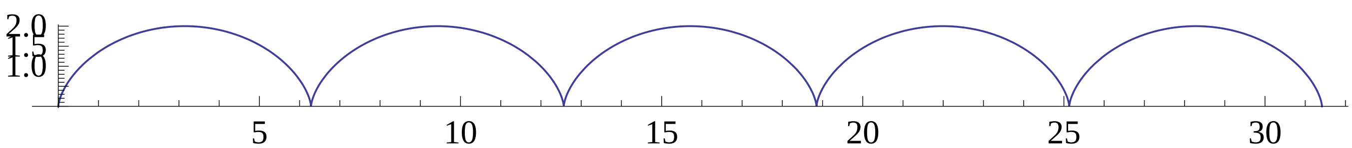


Its parametric equations are given by

$$x = r(t - \sin(t)) \qquad y = r(1 - \cos(t)).$$

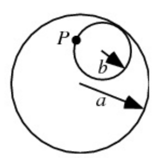
The graph shown below is for $r = 1$ and can be drawn in Mathematica with the following command:

```
In[10]:= ParametricPlot[{t - Sin[t], 1 - Cos[t]},
           {t, 0, 10 * Pi}]
```




## The Hypercycloid

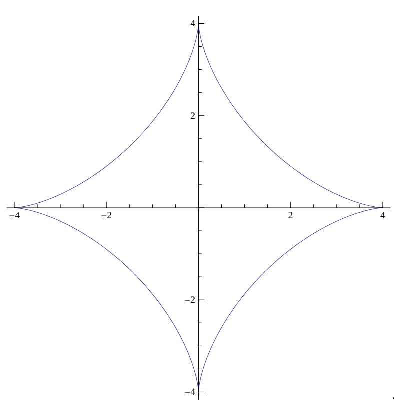
This is a curce traced out by a fixed point $P$ on a circle $C$ of radius $b$ as C rolls around inside a circle with center 0 and radius $a$.




The parametric equations are given by

$$x = (a - b)\cos t + b\cos\left(\frac{a-b}{b}t\right), \quad y = (a - b)\sin t - b\sin\left(\frac{a-b}{b}t\right)$$

If $a = 4$ and $b = 1$, we gat a curve with 4 cusps called an astroid shown on the left below.
If $a = 5$ and $b = 1$, we get a curve with 5 cusps shown in the center below.
If $a = \pi$ and $b = 1$, we get a curve with infinitely many cusps shown on the right below for $0 \leq t \leq 50\pi$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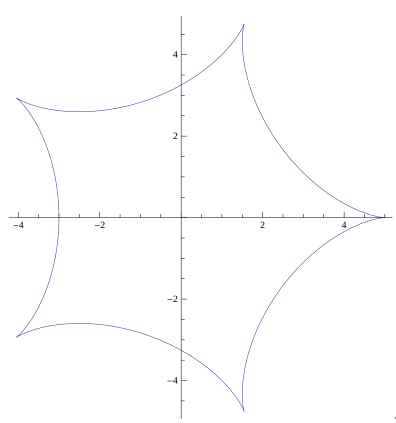
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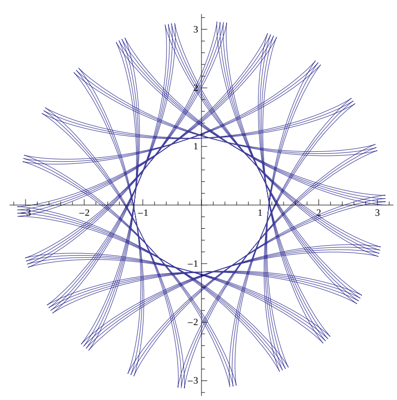


## Spirographs

When you use a spirograph, the pencil is not on the small circle, but somewhere between the center of the small circle and the circle itself. If the pencil is placed at a distance $pb$ away from the center of the inner circle, where $0 < p < 1$, the curve traced out by the pencil is given by

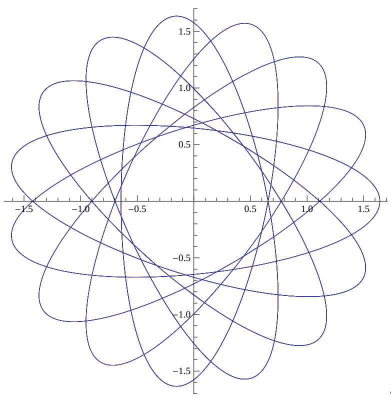
$$x = (a-b)\cos t + pb\cos\left(\frac{a-b}{b}t\right), \quad y = (a-b)\sin t - pb\sin\left(\frac{a-b}{b}t\right)$$

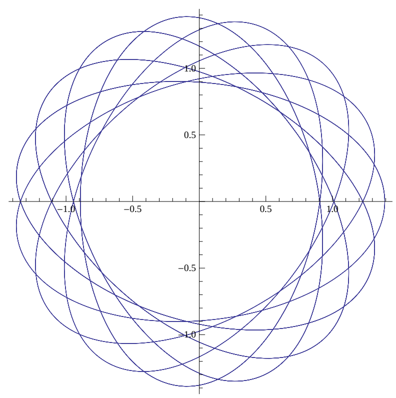
The curves shown below have

$$a = 15/7, b = 1, p = 1/2 \ \ \text{(on left)}$$

and

$$a = 15/7, b = 1, p = 1/4 \ \ \text{(on right)}$$





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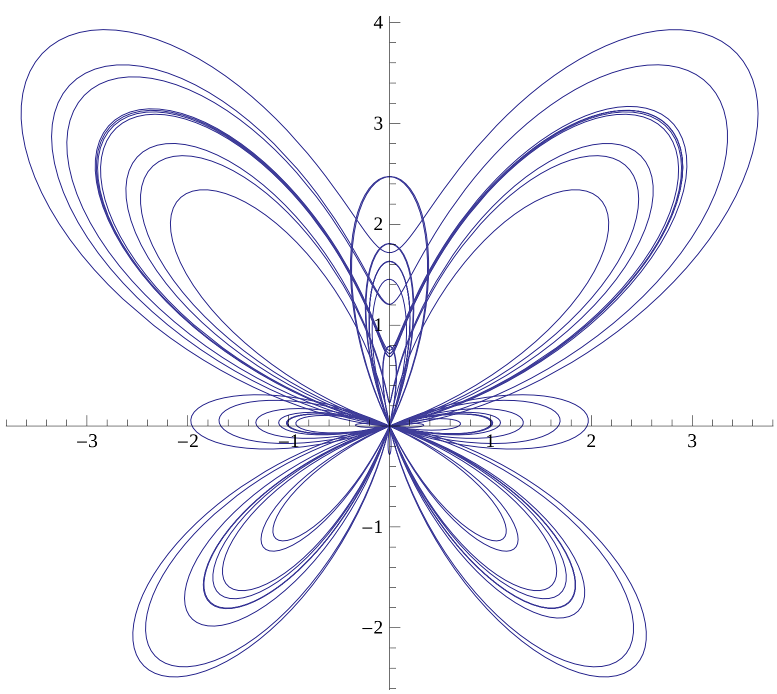
## Butterfly

The following equation gives a butterfly:

$$x = \sin(t)\left[e^{\cos t} - 2\cos(4t) - \sin((1/12)t)^5\right], \quad y = \cos(t)\left[e^{\cos t} - 2\cos(4t) - \sin((1/12)t)^5\right]$$



## Lissajous

A family of curves called Lissajous curves are given by

$$x = a\sin(nt), \quad y = b\cos(mt).$$

Below we show the result with

$$a = 2, \quad n = 10, \quad b = 3, \quad m = 1$$