Lecture 34: Curves Defined by Parametric Equations

When the path of a particle moving in the plane is not the graph of a function, we cannot describe it using a formula that express \( y \) directly in terms of \( x \), or \( x \) directly in terms of \( y \). Instead, we need to use a third variable \( t \), called a parameter and write:

\[
\begin{align*}
x &= f(t) \\
y &= g(t)
\end{align*}
\]

The set of points \((x, y) = (f(t), g(t))\) described by these equations when \( t \) varies in an interval \( I \) form a curve, called a parametric curve, and \( x = f(t), y = g(t) \) are called the parametric equations of the curve. Often, \( t \) represents time and therefore we can think of \((x, y) = (f(t), g(t))\) as the position of a particle at time \( t \).

If \( I \) is a closed interval, \( a \leq t \leq b \), the point \((f(a), g(a))\) is the initial point and the point \((f(b), g(b))\) is the terminal point.

**Example 1** Draw and identify the parametric curve given by the parametric equations:

\[
\begin{align*}
x &= \cos t \\
y &= \sin t \\
0 \leq t \leq 2\pi
\end{align*}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \cos 0 )</td>
<td>( \sin 0 )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \cos \frac{\pi}{4} )</td>
<td>( \sin \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \cos \frac{\pi}{2} )</td>
<td>( \sin \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( \cos \frac{3\pi}{4} )</td>
<td>( \sin \frac{3\pi}{4} )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( \cos \pi )</td>
<td>( \sin \pi )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>( \cos \frac{3\pi}{2} )</td>
<td>( \sin \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>2\pi</td>
<td>( \cos 2\pi )</td>
<td>( \sin 2\pi )</td>
</tr>
</tbody>
</table>

As \( t \) increases, we travel along the curve in a particular direction giving the curve an orientation which is often indicated by arrows. The curve however may pass through a section of the curve repeatedly and may do so in different directions for different values of \( t \).

**Example 2** Describe the parametric curve represented by the parametric equations:

\[
\begin{align*}
x &= \sin 2t \\
y &= \cos 2t \\
0 \leq t \leq 2\pi
\end{align*}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \sin 0 )</td>
<td>( \cos 0 )</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \sin \frac{\pi}{4} )</td>
<td>( \cos \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>( \sin \frac{\pi}{2} )</td>
<td>( \cos \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>( \sin \frac{3\pi}{4} )</td>
<td>( \cos \frac{3\pi}{4} )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( \sin \pi )</td>
<td>( \cos \pi )</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>( \sin \frac{3\pi}{2} )</td>
<td>( \cos \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>2\pi</td>
<td>( \sin 2\pi )</td>
<td>( \cos 2\pi )</td>
</tr>
</tbody>
</table>
Note The curve in examples 1 and 2 are the same but the parametric curve are not. Because in one case the point \((x, y) = (\cos t, \sin t)\) moves once around the circle in the counterclockwise direction starting from \((1, 0)\). In example 2 instead, the point \((x, y) = (\sin 2t, \cos 2t)\) moves twice around the circle in the clockwise direction starting from \((0, 1)\).

**Example 3** Sketch the graph of the curve described by the following set of parametric equations.

\[ x = t^3 - t, \quad y = t^2, \quad 0 \leq t < \infty \]

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Converting from parametric equations to equations in Cartesian co-ordinates.

There is no exact method for converting parametric equations for a curve to an equation in \(x\) and \(y\) only. If we can solve for \(t\) in terms of either \(x\) or \(y\), we can substitute this for the value of \(t\) in one of the equations to get an equation in \(x\) and \(y\) only.

**Example 4** Covert the following parametric equation to an equation relating \(x\) and \(y\):

\[ x = 2t + 1, \quad y = t - 2. \]

and describe the curve traced as \(t\) runs from \(-\infty\) to \(\infty\).

Sometimes, we can see a relationship between the \(x\) and \(y\) co-ordinates and thus eliminate the \(t\).

**Example 5** Convert the following parametric equation to an equation relating \(x\) and \(y\):

\[ x = 2 \cos t \quad y = 3 \sin t \]

and describe the curve traced when \(0 \leq t \leq 4\pi\).
Converting from equations in Cartesian co-ordinates to parametric equations

**Easy cases** If a curve is defined by the equation \( y = f(x) \), the equations \( x = t \) and \( y = f(t) \) give parametric equations describing the curve.

If a curve is described by the equation \( x = g(y) \), the equations \( x = t \) and \( x = g(t) \) give parametric equations describing the curve.

**Example 6** Give parametric equations describing the graph of the parabola \( y = x^2 \).

**Example 7** Find parametric equations on \( 0 \leq t \leq 2\pi \) for the motion of a particle that starts at \((a,0)\) and traces the circle \( x^2 + y^2 = a^2 \) twice counterclockwise.

Extra examples, see computer graphs or plot some points.

**Spiral**

The following equations describe a spiral with an anticlockwise direction:

\[
    x = t \cos t, \quad y = t \sin t.
\]

The graph shown below can be drawn in Mathematica with the following command:

\[
    \text{In[13]= ParametricPlot[\{t*Cos[t], t*Sin[t]\}, \{t, 0, 10*Pi\}]}\]
The cycloid

The curve traced out by a point $P$ on the circumference of a circle of radius $r$ as the circle rolls along a straight line is called a cycloid.

\[
x = r(t - \sin(t)) \quad y = r(1 - \cos(t)).
\]

The graph shown below is for $r = 1$ and can be drawn in Mathematica with the following command:

\[
\text{In[10]} = \text{ParametricPlot}[\{t - \text{Sin}[t], 1 - \text{Cos}[t]\}, \{t, 0, 10 \cdot \text{Pi}\}]
\]

The Hypercycloid

This is a curve traced out by a fixed point $P$ on a circle $C$ of radius $b$ as $C$ rolls around inside a circle with center 0 and radius $a$.

The parametric equations are given by

\[
x = (a - b) \cos t + b \cos \left(\frac{a - b}{b} t\right), \quad y = (a - b) \sin t - b \sin \left(\frac{a - b}{b} t\right)
\]

If $a = 4$ and $b = 1$, we get a curve with 4 cusps called an astroid shown on the left below.
If $a = 5$ and $b = 1$, we get a curve with 5 cusps shown in the center below.
If $a = \pi$ and $b = 1$, we get a curve with infinitely many cusps shown on the right below for $0 \leq t \leq 50\pi$.
**Spirographs**

When you use a spirograph, the pencil is not on the small circle, but somewhere between the center of the small circle and the circle itself. If the pencil is placed at a distance $pb$ away from the center of the inner circle, where $0 < p < 1$, the curve traced out by the pencil is given by

$$x = (a - b) \cos t + pb \cos \left(\frac{a - b}{b} t\right), \quad y = (a - b) \sin t - pb \sin \left(\frac{a - b}{b} t\right)$$

The curves shown below have

$$a = 15/7, b = 1, p = 1/2 \quad (\text{on left})$$

and

$$a = 15/7, b = 1, p = 1/4 \quad (\text{on right})$$

![Spirograph Curves](image)

**Butterfly**

The following equation gives a butterfly:

$$x = \sin(t)\left[e^{\cos t} - 2 \cos(4t) - \sin((1/12)t)^5\right], \quad y = \cos(t)\left[e^{\cos t} - 2 \cos(4t) - \sin((1/12)t)^5\right]$$

![Butterfly](image)
A family of curves called Lissajous curves are given by

\[ x = a \sin(nt), \quad y = b \cos(mt). \]

Below we show the result with

\[ a = 2, \quad n = 10, \quad b = 3, \quad m = 1 \]